ROBUST CONTROLLERS DESIGN FOR SISO SYSTEMS

Mária HYPIUSOVÁ and Ľubomír GRMAN

Institute of Control and Industrial Informatics,
Slovak University of Technology, Faculty of Electrical Engineering and Information Technology,
Ilkovičova 3, 812 19 Bratislava 1, Slovak Republic
E-mail: maria.hypiusova@stuba.sk

Abstract

The paper deals with the design of robust controllers for uncertain SISO systems in the frequency domain. The controller designed using the Kharitonov systems and Bode characteristics guarantees the required phase margin, and the controllers designed using the Edge Theorem and Small Gain Theorem guarantee the required degree of stability. The practical application of the three approaches is illustrated by the robust controller design for a DC-motor.

Keywords: Robust control, Kharitonov‘s Theorem, Edge Theorem, Small Gain Theorem

1 INTRODUCTION

For many real processes a controller design has to cope with the effect of uncertainties, which very often cause a poor performance or even instability of closed-loop systems. The reason for that is a perpetual time change of parameters (due to aging, influence of environment, working point changes *etc.*), as well as unmodelled dynamics. The former uncertainty type is denoted as the parametric uncertainty and the latter one the dynamic uncertainty. A controller ensuring closed-loop stability under both of these uncertainty types is called a robust controller. A lot of robust controller design methods are known from the literature (Ackermann, 1997; Bhattacharyya, Chapellat and Keel, 1995) in the time- as well as in the frequency domains.

In this paper three approaches to robust controller design have been applied for three working points of a DC-motor. The first method is based on the Kharitonov systems and Bode diagram for interval model. This controller guarantees the required phase margin. The second approach is accomplished with the Edge Theorem and the Neymark D-partition method for the affine model. The last controller design method is based on the Small Gain Theorem considering uncertain system model with additive uncertainty. For the second and the third methods the designer can specify a required closed-loop stability degree.

2 PRELIMINARIES AND PROBLEM FORMULATION

2.1 Robust controller design using the Kharitonov system

Consider three working points of the controlled system. Comparing coefficients of equal powers of s of the three transfer functions we obtain intervals of coefficients:

$$b_i \in \left\langle \underline{b}_i, \overline{b}_i \right\rangle \quad \text{and} \quad a_j \in \left\langle \underline{a}_j, \overline{a}_j \right\rangle \quad \text{for } i = 0,1,m;\ j = 1,2,\dots,n \tag{1}$$

where $a_n \neq 0; b_m \neq 0; m \leq n$.

Interval polynomials of the system numerator and denumerator respectively are given

$$\mathbf{N}(s) = \left\{ B(s) : b_0 + \ldots + b_m s^m , b_i \in \left\langle \underline{b}_i , \overline{b}_i \right\rangle, i = 0,1,\ldots,m \right\} \tag{2}$$

$$\mathbf{D}(s) = \left\{ A(s) : a_0 + \ldots + a_n s^n , a_i \in \left\langle \underline{a}_i , \overline{a}_i \right\rangle, i = 0,1,\ldots,n \right\} \tag{3}$$

The interval system is defined in the form

$$\mathbf{G}(s) = \left\{ \frac{B(s)}{A(s)} : (B(s), A(s)) \in (\mathbf{N}(s), \mathbf{D}(s)) \right\} \tag{4}$$

Consider $I(s)$ to be the set of closed-loop characteristic polynomials of degree n for the interval systems (4)

$$p(s) = A(s) + B(s) = p_0 + p_1 s + p_2 s^2 + \ldots + p_n s^n \tag{5}$$

where $p_0 \in \left\langle \underline{p}_0, \overline{p}_0 \right\rangle$, $p_1 \in \left\langle \underline{p}_1, \overline{p}_1 \right\rangle$, ..., $p_n \in \left\langle \underline{p}_n, \overline{p}_n \right\rangle$.
Such a set of polynomials is called *interval family* and we refer to $I(s)$ as to interval polynomial. The necessary and sufficient condition for the stability of the entire family is formulated in the Kharitonov's Theorem.

Theorem 1 (Kharitonov's Theorem)
Every polynomial in the family $I(s)$ is stable if and only if the following four extreme polynomials are stable:

$$\begin{aligned}
K^1(s) &= \underline{p}_0 + \underline{p}_1 s + \overline{p}_2 s^2 + \overline{p}_3 s^3 + \underline{p}_4 s^4 + \ldots = p^{--} \\
K^2(s) &= \underline{p}_0 + \overline{p}_1 s + \overline{p}_2 s^2 + \underline{p}_3 s^3 + \underline{p}_4 s^4 + \ldots = p^{-+} \\
K^3(s) &= \overline{p}_0 + \underline{p}_1 s + \underline{p}_2 s^2 + \overline{p}_3 s^3 + \overline{p}_4 s^4 + \ldots = p^{+-} \\
K^4(s) &= \overline{p}_0 + \overline{p}_1 s + \underline{p}_2 s^2 + \underline{p}_3 s^3 + \overline{p}_4 s^4 + \ldots = p^{++}
\end{aligned} \tag{6}$$

□

If the coefficients p_i vary dependently, the Kharitonov's Theorem is conservative. In such a case we can use the set of Kharitonov systems as follows:

$$\mathbf{G}_K(s) = \left\{ \frac{K_B^i(s)}{K_A^j(s)} : i, j = 1,2,3,4 \right\} \tag{7}$$

where $K_B^i(s)$, i = 1,2,3,4 and $K_A^j(s)$, j = 1,2,3,4 denote the Kharitonov polynomials associated with $\mathbf{N}(s)$ and $\mathbf{D}(s)$ respectively.

Theorem 2
The closed loop system containing the interval plant **G**(s) is robustly stable if and only if each of the Kharitonov systems in $\mathbf{G}_K(s)$ is stable. □

It means that the controller is to be designed for the worst Kharitonov systems (with the least phase margin). Then standard methods can applied to the controller design, e.g. Bode characteristics with guaranteeing the required phase margin.

2.2 Robust controller design using the Edge Theorem

If a part of coefficients of the plant vary dependently, then it is better to use the affine model of the plant in the form:

$$G(s)=\frac{B(s)}{A(s)}=\frac{b_0(s)+\sum_{i=1}^{p} b_i(s)q_i}{a_0(s)+\sum_{i=1}^{p} a_i(s)q_i} \tag{8}$$

where $q_i \in \langle \underline{q}_i, \overline{q}_i \rangle$ are uncertain coefficients. The coefficients depend linearly on uncertain parameter vector $\mathbf{q}^T = [q_1,...,q_p]$. The parameters q_i vary within a p - dimensional box

$$\mathbf{Q}=\{\mathbf{q}: q_i \in \langle \underline{q}_i, \overline{q}_i \rangle, i=1,...,p\}. \tag{9}$$

If we vary parameters $q_i = \underline{q}_i$ or $q_i = \overline{q}_i$ then is possible to obtain 2^p transfer functions with constant coefficients; inserting them to the vertices of a p - dimensional polytope, the transfer function (8) describes a so-called *polytopic system*. Consider the controller transfer function in the form

$$G_R(s)=\frac{F_1(s)}{F_2(s)} \tag{10}$$

where $F_1(s)$ and $F_2(s)$ are polynomials with constant coefficients. Then the characteristic polynomials with the polytopic system are

$$p(s,\mathbf{q})=b_0(s)F_1(s)+a_0(s)F_2(s)+\sum_{i=1}^{p} q_i\left[b_i(s)F_1(s)+a_i(s)F_2(s)\right] \tag{11}$$

or in a more general form

$$p(s,\mathbf{q})=p_0(s)+\sum_{i=1}^{p} q_i p_i(s) \qquad q_i \in \mathbf{Q} \tag{12}$$

Theorem 3 (Edge Theorem)
The polynomial family (12) is stable if and only if the edges of **Q** are stable. □

The simple stability analysis method for families of polynomials (edges of **Q**) is given in the following theorem.

Theorem 4 (Bialas)
Let $H_n^{(a)}$ and $H_n^{(b)}$ be the Hurwitz matrices of

$$\begin{aligned} p_b(s) &= p_{b0}+p_{b1}s+p_{b2}s^2+...+p_{bn}s^n \quad p_{bn}>0, \\ p_a(s) &= p_{a0}+p_{a1}s+p_{a2}s^2+...+p_{an}s^n \quad p_{an}>0, \end{aligned} \tag{13}$$

respectively. The polynomial family

$$p(s,Q)=\{\lambda p_a(s)+(1-\lambda)p_b(s), \quad \lambda \in [0,1]\} \tag{14}$$

is stable if and only if:

1) $p_b(s)$ is stable

2) the matrix $\left(H_n^{(b)}\right)^{-1} H_n^{(a)}$ has no nonpositive real eigenvalues. □

Using the Edge Theorem, the controller design has to be applied to 4 vertices of the polytopic system; by applying e.g. the Neymark's D-partition method guaranteeing the required closed-loop stability degree we choose the controller coefficients such that the vertices of polytopic system are stable. Then we have to check stability of each edge of the box **Q** by e.g. the Bialas Theorem. If any of the edges is unstable, new controller coefficients are to be chosen by Neymark's method.

2.3 Robust controller design using the Small Gain Theorem

Consider a perturbed plant with unstructured additive uncertainties in the form

$$G_p(s) = G_{nom}(s) + \partial G(s) \tag{15}$$

where $G_{nom}(s)$ is the nominal model and $\partial G(s)$ are additive uncertainties.
The nominal model can be obtained e.g. by N identifications of the plant (in N working points) by taking mean values of the nominator and denominator coefficients, respectively:

$$G_{nom}(s) = \frac{(B_1(s) + \ldots + B_N(s))/N}{(A_1(s) + \ldots + A_N(s))/N} \tag{16}$$

For each ω the uncertainties are found by substituting $s = j\omega - \alpha$ where α is the required stability degree:

$$\delta G(\omega) = \max\left|G_{nom}(s) - G_{p_i}(s)\right|_{s=j\omega-\alpha} \qquad \text{for } i = 1,\ldots,N \tag{17}$$

Theorem 5 (Small Gain Theorem)

Assume that the open-loop system is stable. The closed-loop system is stable if and only if the open-loop magnitude satisfies

$$\left|G_R(j\omega)G_p(j\omega)\right| < 1 \qquad \text{for } \omega \in \langle 0,\infty\rangle \tag{18}$$

□

Theorem 6

Consider an auxiliary characteristic polynomial in the form

$$1 + F_{URO}(s)\frac{\delta G(s)}{G_{nom}(s)} \tag{19}$$

where $F_{URO}(s) = \dfrac{G_{nom}(s)G_R(s)}{1 + G_{nom}(s)G_R(s)}$. (20)

Assume that the open-loop system (nominal model and controller) and the auxiliary characteristic polynomial (19) are stable. Then closed-loop characteristic polynomial $p(s)=1+G_p(s)G_R(s)$ with unstructured additive uncertainties (15) is stable if and only if the following condition holds:

$$\left|F_{URO}(j\omega-\alpha)\right| < \frac{1}{\left|\dfrac{\delta G(\omega)}{G_{nom}(j\omega-\alpha)}\right|} = M_0(\omega) \qquad \text{for } \omega\in\langle 0,\infty\rangle \tag{21}$$

□

The condition (21) is verified grafically. The robust controller design using Small Gain Theorem is realized according to the following steps:

1. Specify the closed – loop system magnitude corresponding to the transfer function:

$$W(s)=\frac{G_{nom}(s)G_R(s)}{1+G_{nom}(s)G_R(s)} \tag{22}$$

   If the nominal model is of the second order then $W(s)=\dfrac{as+1}{bs+1}$

   and $G_R(s)=\dfrac{W(s)}{G_{nom}(s)-W(s)G_{nom}(s)}$ is a PID controller.

2. Choose the numerator of $W(s)$ equal to the numerator of G_{nom}
3. Choose $b>a$ and design the robust controller so that (21) is satisfied.

3 EXAMPLE

Consider the transfer functions of a DC motor obtained by identification in three working points:

WP1: manipulated variable $u=7.5$ [V]; regulated variable $y=3650$ [1/min];
load $z=7$ [V]

$$G_{p_1}(s)=\frac{-0.483s+5.117}{s^2+3.112s+3.045}$$

WP2: $u=5$ [V]; $y=1550$ [1/min]; $z=7$ [V]

$$G_{p_2}(s)=\frac{-0.352s+3.904}{s^2+2.637s+2.353}$$

WP3: $u=6.5$ [V]; $y=2250$ [1/min]; $z=3$ [V]

$$G_{p_3}(s)=\frac{-0.267s+2.465}{s^2+2.387s+1.662}$$

Robust controller design using Kharitonov systems

The Kharitonov approach uses the interval model:

$$G_p(s)=\frac{B(s)}{A(s)}=\frac{b_1s+b_0}{a_2s^2+a_1s+1}$$

where $b_0 \in \langle 1.484, 1.681 \rangle$, $b_1 \in \langle -0.161, -0.149 \rangle$, $a_1 \in \langle 1.022, 1.436 \rangle$, $a_2 \in \langle 0.328, 0.602 \rangle$.

The required phase margin: $\Delta\varphi_z = 50[°]$. The phase added by PI controller: $\varphi_r = 25[°]$. We have designed a robust PI controller for the worst Kharitonov system (with the least phase margin) in the form

$$R(s) = K\left(1 + \frac{1}{T_i s}\right)$$

where the gain $K = 0.773$ and the integration time constant $T_i = 1.528[s]$.

Robust controller design using the Edge Theorem

The Edge Theorem based approach uses the polytopic model:

$$G_p(s) = \frac{b_0(s) + b_1(s)q_1 + b_2(s)q_2}{a_0(s) + a_1(s)q_1 + a_2(s)q_2}$$

where: $b_0(s) = -0.417s + 4.511$, $a_0(s) = s^2 + 2.874s + 2.699$,

$b_1(s) = -0.042s + 0.719$, $a_1(s) = 0.125s + 0.346$,

$b_2(s) = 0.108s - 1.326$, $a_2(s) = -0.363s - 0.691$

q_i - uncertain coefficients

The required degree of stability: $\alpha = 0$

The robust PID controller has been designed by Neymark's D-partition method for 4 vertices of the polytopic system

$$G_R(s) = K\left(1 + \frac{1}{T_i s} + T_d s\right)$$

where the gain $K = 2$, the integration time constant $T_i = 1.333\ [s]$ and the derivative time constant $T_d = 0.5\ [s]$. Stability has been verified for each edge of the box **Q** by Bialas Theorem and all eigenvalues of the Bialas matrices were not nonpositive real. Therefore, the closed-loop with polytopic systems and robust controller is stable and the achieved degree in 4 vertices is $\alpha = 0.872$.

Robust controller design using the Small Gain Theorem

The Small Gain Theorem based approach uses the uncertain model with additive uncertainties. The nominal model: $G_{nom} = \dfrac{-0.3749s + 3.791}{s^2 + 2.75s + 2.353}$

and uncertainties for the required degree of $\alpha = 0$ are depicted in Fig. 1. The desired transfer function: $W(s) = \dfrac{-0.0989s + 1}{bs + 1}$. We have designed two robust PID controllers:

$b = 0,5$ $K = 1.211$, $T_i = 1.168[s]$, $T_d = 0.364[s]$

$b = 1$ $K = 0.66$, $T_i = 1.168[s]$, $T_d = 0.364[s]$

Condition (21) was satisfied for each design.

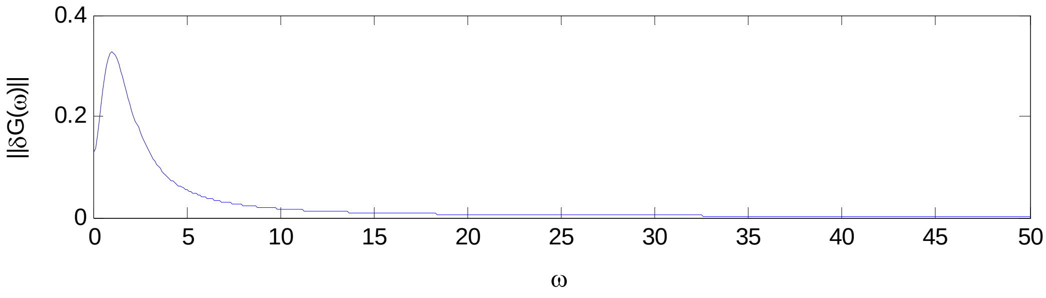

Figure 1: $\delta G(\omega)$ - versus – ω plot

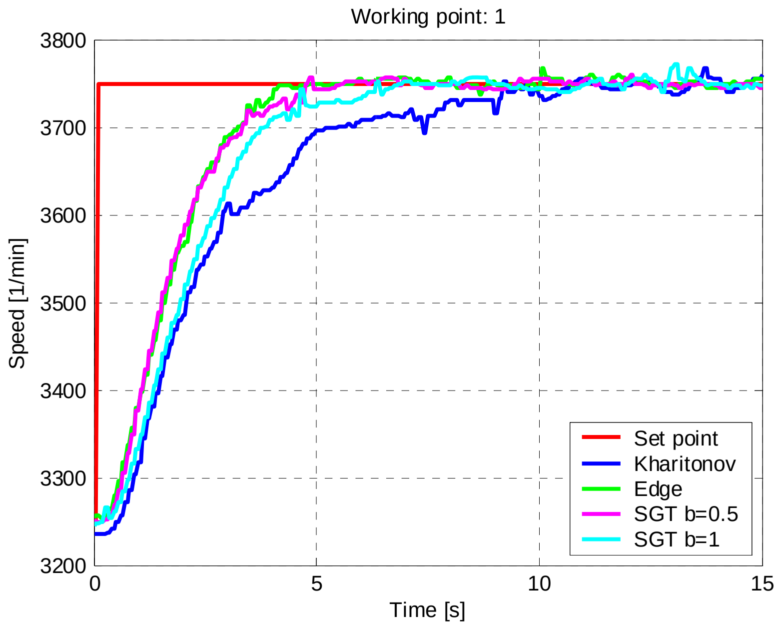

Figure 2: Step responses in the first working point

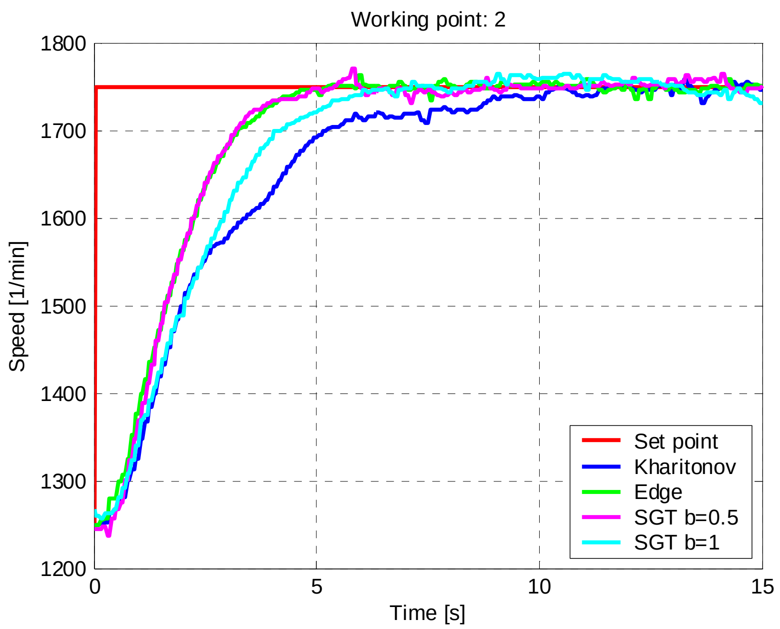

Figure 3: Step responses in the second working point

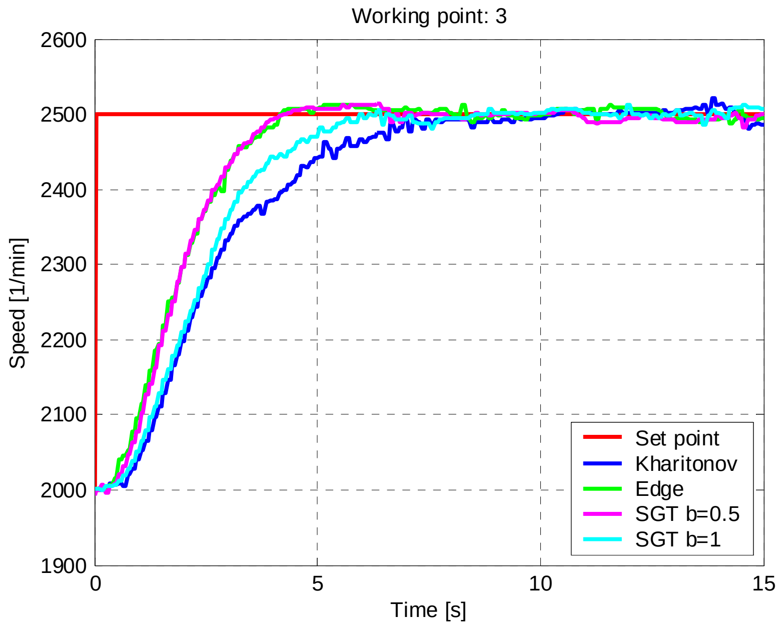

Figure 4: Step responses in the third working point

Fig. 2, 3 and 4 show the step responses designed robust controller in three working points.

4 CONCLUSION

The main aim of this paper has been to apply three methods to design a robust controller for a DC motor. The best approaches were the Edge Theorem based method and the Small Gain Theorem based method. These methods guarantee the required closed-loop stability degree.

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