CONTROL OF SWITCHED LINEAR CONTINUOUS-TIME SYSTEMS: MULTIPLE LYAPUNOV FUNCTION

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Abstract: The paper addresses the problem output feedback controller design for switched linear continuous-time systems by multiple Lyapunov function quadratic stability. The method also determines a minimum dwell time. Numerical examples are given to illustrate the design procedure.

Keywords: Switched systems, Linear matrix inequality, Multiple Lyapunov function, Output feedback.

1 INTRODUCTION

Switched systems have attracted considerable attention of many researchers in the world during the past decades. A switched system has hybrid dynamic features comprising a family of subsystems described by continuous-time or discrete-time dynamics, and a rule specifying the switching among them. The studies on such systems are motivated by the fact that many physical systems and man-made systems are often modelled based on such a framework exhibiting switching features [Imura, 2003], [Johansson, 1998], [Rantzer, 2000]. Publications on the general topic and recent research progress in the field of switched systems are e.g. [DeCarlo, 2000], [Liberzon, 1999], [Sun, 2005]. In the last years, a hot topic is to find less conservative conditions to guarantee the stability of switched systems under arbitrary switching signals [Branicky, 1998], [Mignone, 2000], [Ye, 1998].

In this paper, the problem of switched linear continuous-time systems control using the multiple Lyapunov function is considered.

2 PROBLEM FORMULATION AND PRELIMINARIES

This paper is concerned with continuous-time switched linear systems of the following general form

$$\dot{x}(t) = A_{c\sigma(t)}x(t), \qquad x(0) = x_0 \tag{1}$$

defined for all $t \geq 0$ where $x(t) \in R^n$ is the state, $\sigma(t)$ is the switching rule and x_0 is the initial condition. We consider the class of switched systems characterized by the fact that for each $t \geq 0$, the switching rule is such that

$$A_{c\sigma(t)} \in \{A_{c1}, A_{c2}, \ldots, A_{cN}\} \tag{2}$$

The model (2) naturally imposes a discontinuity on $A_{c\sigma(t)}$ since this matrix must jump instantaneously from A_i to A_j for some $i \neq j = 1, \ldots, N$ once switching occurs. In other words, $A_{c\sigma(t)}$ is constrained to jump among the *N* vertices of the matrix polytope $\{A_{c1}, A_{c2}, \ldots, A_{cN}\}$.

Lemma 1. (Quadratic stability)

A switched system (1) is quadratically stable if and only if there exist a positive definite matrix $P = P^T > 0$ such that

$$A_{ci}^T P + P A_{ci} < 0 \qquad i = 1,2,\ldots,N \tag{3}$$

□

Unfortunately this approach generally provides quite conservative results. To reduce the conservatism of (3) the multiple Lyapunov function has been proposed [Geromel, 2005]. When $\sigma(t)$ is a piecewise constant signal, the stability conditions can be obtained using a family of symmetric and positive definite matrices $\{P_1,\ldots,P_N\}$ each of them associated to each matrix of the set $\{A_{c1},\ldots,A_{cN}\}$. If the Lyapunov function is non-increasing in time with respect to $\sigma(t)$ at every switching time, then the global system is stable.

Lemma 2.

Assume that, for some $T > 0$, there exists a collection of positive definite matrices $\{P_1,\ldots,P_N\}$ of compatible dimensions such that

$$A_{ci}^T P_i + P_i A_{ci} < 0 \qquad \forall\, i = 1,\ldots,N \tag{4}$$

$$e^{A_{ci}^T T} P_j e^{A_{ci} T} - P_i < 0 \qquad \forall\, i \neq j = 1,\ldots,N \tag{5}$$

where T is dwell-time of the switched system; t_k and t_{k+1} are successive switching times satisfying $t_{k+1} - t_k \geq T$ for all k. The time switching control

$$\sigma(t) = i \in \{1,\ldots,N\} \qquad t \in [t_k, t_{k+1}) \tag{6}$$

makes the equilibrium solution $x = 0$ of (1) globally asymptotically stable. □

3 CONTROL DESIGN

Assume the open loop switched system (1) is as follows

$$\begin{aligned} \dot{x}(t) &= A_i x(t) + B_i u(t) \\ y(t) &= C_i x(t) \qquad i = 1,2,\ldots,N \end{aligned} \tag{7}$$

The problem studied in this paper can be formulated as follows: For switched systems described by (7) a static output feedback controller is to be designed with the gain matrix F and control algorithm

$$u(t) = F y(t) = F C_i x(t) \tag{8}$$

such that the closed loop system

$$\dot{x}(t) = (A_i + B_i F C_i) x(t) = A_{ci} x(t) \tag{9}$$

is stable with respect to multiple Lyapunov function and guaranteed cost.

Definition 1.

Consider the switched system (7) and the control algorithm (8), If there exists a control law u^* and a positive scalar J^* such that the closed loop system (9) is stable and the cost function

$$J = \int_0^\infty \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right]dt \tag{10}$$

satisfies $J \le J^*$ for $i = 1,2,\ldots,N$ then J^* is said to be guaranteed cost and u^* is said to be the guaranteed cost control law for system (7). □

Lemma 3.

For the switched system (7) the control algorithm (8) ensures the guaranteed cost if the following two conditions hold

$$\begin{aligned} &A_{ci}^T P_i + P_i A_{ci} + Q + C^T F^T RFC < 0 \qquad \forall\, i = 1,\ldots,N \\ &e^{A_{ci}^T T} P_j e^{A_{ci} T} - P_i < 0 \qquad \forall\, i \ne j = 1,\ldots,N \end{aligned} \tag{11}$$

□

The above two inequalities are the basis for the design procedure developed below. Inequalities (11) can be extended and modified to the form

$$(A_i + B_i FC)^T P_i + P_i (A_i + B_i FC) + Q + C^T F^T RFC < 0\,, \qquad i = 1,2,\ldots,N \tag{12}$$

$$e^{(A_i + B_i FC)^T T} P_j e^{(A_i + B_i FC)T} - P_i < 0\,, \qquad i \ne j = 1,2,\ldots,N \tag{13}$$

In the system of inequalities (12) and (13) the positive definite matrix P_i and the feedback gain F are unknown. The N inequalities (12) can be modified to the following quadratic matrix inequalities (QMIs)

$$A_i^T P_i + P_i A_i + Q - P_i B_i R^{-1} B_i^T P_i + \left(FC + R^{-1} B_i^T P_i\right)^T R \left(FC + R^{-1} B_i^T P_i\right) < 0\,, \tag{14}$$

If it is possible to find $P_i > 0$ and F satisfying the QMI in (14), then a stabilizing static output feedback gain exists.

Due to the negative sign in the $-P_i B_i R^{-1} B_i^T P_i$ term, LMI cannot be applied to (14). To accommodate the $-P_i B_i R^{-1} B_i^T P_i$ term, we introduce an additional design variable X. By linearization [Han, 2003] using the inequality $(X_i - P_i)^T B_i R^{-1} B_i^T (X_i - P_i) \ge 0$ for any X and P of the same dimension, we obtain

$$X_i^T B_i R^{-1} B_i^T P_i + P_i^T B_i R^{-1} B_i^T X_i - X_i^T B_i R^{-1} B_i^T X_i \le P_i^T B_i R^{-1} B_i^T P_i\,, \tag{15}$$

with equality sign for $X_i = P_i$. By combining (15) and (14) we obtain a sufficient condition for the existence of static output feedback matrix F given by

$$\begin{aligned} &A_i^T P_i + P_i A_i + Q - X_i B_i R^{-1} B_i^T P_i - P_i B_i R^{-1} B_i^T X_i + X_i B_i R^{-1} B_i^T X_i + \\ &+ \left(FC + R^{-1} B_i^T P_i\right)^T R \left(FC + R^{-1} B_i^T P_i\right) < 0 \end{aligned} \quad \text{for } i = 1,2,\ldots,N \tag{16}$$

Using the Schur complement, inequalities (16) for fixed matrix X are equivalent to the following LMIs

$$\begin{bmatrix} A_i^T P_i + P_i A_i + Q - X_i B_i R^{-1} B_i^T P_i - P_i B_i R^{-1} B_i^T X_i + X_i B_i R^{-1} B_i^T X_i & \left(FC + R^{-1} B_i^T P_i\right)^T \\ FC + R^{-1} B_i^T P_i & -R^{-1} \end{bmatrix} < 0 \qquad (17)$$

These LMIs can be solved iteratively. The LMI problem is convex and can be efficiently solved if a feasible solution exists.

Algorithm

1. Select $Q = Q^T > 0$, $R = R^T > 0$, $F_0 = 0$, $T > 0$ and choose the initial value of X , e.g. from the following algebraic Riccati equation

$$A_i^T P_i + P_i A_i - P_i B_i R^{-1} B_i^T P_i + Q = 0\,, \qquad i = 1,2,\ldots,N \qquad (18)$$

   Set $k = 1$ and $X_i = P_i$.
2. For the matrices X_i are known, compute F_k and $P_i > 0$ using the matrix inequalities (17).
3. For the known $F_{k\text{-}1}$ compute P_i and P_j using the matrix inequalities (13).
4. Compute $er = \|X_i - P_i\|$.

   If $er \le tolerance$ stop, else $k = k + 1$, $X_i = P_i$ and go to Step 2.

If the algorithm fails to arrive at a stabilizing solution, we may select another Q and run the LMI algorithm again.

The minimum value of the dwell time T can be calculated with no big difficulty from the optimal solution of the optimization problem

$$\min_{T>0, P_1>0,\ldots,P_N>0} \{T : \ (4),\ (5)\} \qquad (19)$$

Which, for each fixed $T > 0$, reduces to a convex programming problem with LMIs.

4 EXAMPLES

Example 1. The problem is to design two PI controllers for switched linear continuous-time MIMO system by multiple Lyapunov function. The system model is given by (9), where $i = 1,2,3$.

$$A_1 = \begin{bmatrix} 0 & -0.2023 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -0.9341 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.1783 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -0.6504 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.1671 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -0.7780 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.2380 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -0.8335 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0.2617 & 0 \\ 0.1125 & 0 \\ 0 & -0.0585 \\ 0 & -0.0510 \\ -0.0736 & 0 \\ -0.0329 & 0 \\ 0 & 0.3546 \\ 0 & 0.2309 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & -0.1773 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -0.8153 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.1550 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -0.5889 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.1983 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -0.8911 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.1512 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -0.5820 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0.2429 & 0 \\ 0.1426 & 0 \\ 0 & -0.0546 \\ 0 & -0.0570 \\ -0.0979 & 0 \\ -0.0389 & 0 \\ 0 & 0.2266 \\ 0 & 0.2687 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & -0.2273 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1.0942 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.3426 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1.1710 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.1607 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -0.8494 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.2177 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -0.8167 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0.3680 & 0 \\ -0.0169 & 0 \\ 0 & -0.1471 \\ 0 & 0.0328 \\ -0.0946 & 0 \\ -0.0246 & 0 \\ 0 & 0.3805 \\ 0 & 0.2588 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C_1 = C_2 = C_3 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The decentralized control structure for the two PI controllers can be obtained by the choice of the static output feedback gain matrix F structure. It is given as follows

$$F = \begin{bmatrix} f_{11} & 0 & f_{13} & 0 \\ 0 & f_{22} & 0 & f_{24} \end{bmatrix}$$

The static output feedback gain matrices F for $\rho = 100$, $r = 1$, $R = rI$, $q = 0.001$, $Q = qI$ and dwell time $T = 3$ are as follows

$$F = \begin{bmatrix} -3.562 & 0 & -0.379 & 0 \\ 0 & -7.287 & 0 & -0.949 \end{bmatrix}$$

Maximal closed loop eigenvalue is -0.0924.

Example 2. Consider the switched MIMO system from Example 1. We have calculated the minimum dwell time approach to zero.

5 CONCLUSION

In this paper, we have proposed a procedure to design switched linear continuous-time system control using multiple Lyapunov function. The design procedure is based on the stability analysis results of switched systems [Geromel, 2005] belonging to BMI problems. We have reduced the BMI problem to an LMI one using the linearization approach. The results obtained by verification on examples show effectiveness of the proposed method.

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