Journal of ELECTRICAL ENGINEERING, VOL. 57, NO. 6, 2006, 1–8

ROBUST CONTROL OF LINEAR SYSTEMS IN THE FREQUENCY DOMAIN

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In this paper a novel design technique is proposed to guarantee a required performance of the full system by applying the independent design to equivalent subsystems.

Keywords:

1 INTRODUCTION

During the last decades robustness has been recognized as a key issue in the analysis and design of control systems. The history of robust control design based on small-gain-like robustness condition started developing with the pioneering work of Zames where robust control design problem has been formulated as an optimization problem. Only at the end of the 1980's was found a practical solution to this problem. It is worth to mention some algebraic approaches which followed the seminal works of Kharitonov [khar], Bhattacharyya et al [bhata] and Barlet et al [barlet].

In this paper we focus our attention on two robust design problem. First the problem of robust stabilization of an uncertain single input-single output (SISO) plant described by the transfer function with linear or multilinear interval systems is considered. Multiple-input-multiple output (MIMO) systems usually arise as an interconnection of a finite number of subsystems. In case of such systems practical reasons often make restrictions on controller structure necessary or reasonable. The controller split into several local feedbacks becomes a decentralized controller. With the come up of robust frequency domain approach in the 80's several practice oriented techniques were developed, see [sko],[kozv]. The decentralized controller design comprises two steps: 1. selection of control configuration, 2. design of local controllers. In the second part of this paper we deal with the Step 2. The independent design approach has been adopted. In the independent design used in sequel local controllers are designed without considering interactions with other subsystems. In this paper a novel design technique is proposed to guarantee a required performance of the full system by applying the independent design to the equivalent subsystems.

Preliminaries and Model Uncertainties

Consider a closed-loop system comprising the transfer function matrix of the plant $G(s) \in R^{m\times m}$ and the controller $R(s) \in R^{m\times m}$ in the standard feedback configuration,

2 PRELIMINARIES AND MODEL UNCERTAINTIES

Consider a closed-loop system comprising the transfer function matrix of the plant $G(s) \in R^{m\times m}$ and the controller $R(s) \in R^{m\times m}$ in the standard feedback configuration, Fig. 1, where w, u, y, e are respectively vectors of reference, control input, output and control error of compatible dimensions.

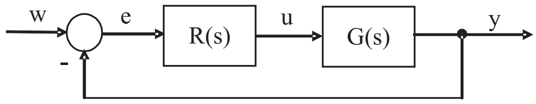

Fig. 1. Standard feedback configuration

The problem addressed in this paper is the design of a robust decentralized controller

$$R(s) = diag\{R_{ii}(s)\}_{m\times m} \tag{1}$$

that guarantees closed-loop stability and performance over the entire operating range of the controlled plant $G(s)$.

Let the plant be given by a set of N transfer function matrices identified in different working points

$$G^k(s) = \{G^k_{ij}(s)\}_{m\times m}, \quad k = 1,2,...N$$
$$\text{with} \quad G^k_{ij}(s) = \frac{y^k_i(s)}{u^k_j(s)} \quad i,j = 1,2,...,m,$$

where $y^k_i(s)$ is the i-th output and $u^k_j(s)$ is the j-th plant input in the $k\ th$ experiment.

Uncertainty associated with a real system model can be described in various ways. There are following types of uncertainty models encountered in the literature: (i) structured uncertainty (parametric uncertainty: interval

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ISSN 1335-3632

model, affine model, multilinear and nonlinear ; dynamic uncertainty with known structure), and (ii) unstructured uncertainty (additive, multiplicative and inverse uncertainty models which include parametric and dynamic uncertainty with unknown structure).

In practice, at a particular frequency the transfer function magnitude and phase are supposed to lie within a disc-shaped region around the nominal transfer function $G_N(s)$. Over a given frequency range, these disc-shaped regions can be generated by the following forms: additive (2), multiplicative input (3) and multiplicative output (4) uncertainties, as well as by the inverse forms [sko]. In the sequel just the first three uncertainty forms will be considered, respectively to describe the uncertain plant $G(s)$.

$$\begin{aligned}\Pi_a: \quad G(s) &= G_N(s) + l_a(s)\Delta(s)\\ l_a(s) &= \max_k \sigma_M[G^k(s) - G_N(s)]\end{aligned} \tag{2}$$

$$\begin{aligned}\Pi_i: \quad G(s) &= G_N(s)[I + l_i(s)\Delta(s)]\\ l_i(s) &= \max_k \sigma_M\{G_N(s)^{-1}[G^k(s) - G_N(s)]\}\end{aligned} \tag{3}$$

$$\begin{aligned}\Pi_o: \quad G(s) &= [I + l_o(s)\Delta(s)]G_N(s)\\ l_o(s) &= \max_k \sigma_M\{[G^k(s) - G_N(s)]G_N(s)^{-1}\}\end{aligned} \tag{4}$$

where $\sigma_M(.)$ is the maximum singular value of the corresponding matrix; $\Delta(s)$ is the uncertainty matrix that satisfies

$$\Delta(s)^T\Delta(s) \leq I \tag{5}$$

For the SISO case, $m = 1$, the multiplicative input and output uncertainties equal. In the frequency domain, uncertain SISO systems can be described using either of the above uncertainty types as well as the following ones :
-linear interval,(linear) affine, multilinear and nonlinear uncertainties.
Consider a SISO plant ($m = 1$) and a controller with transfer functions in the following forms

$$G(s) = \frac{P_1(s)}{P_2(s)} \quad R(s) = \frac{R_1(s)}{R_2(s)} \tag{6}$$

where $P_i(s), i = 1,2$ are a linear interval polynomials

$$P_i(s) = p_{oi} + p_{1i}s + ... + p_{n_i i}s^{n_i} \tag{7}$$

with $p_{ji} \in < \underline{p_{ji}}, \overline{p_{ji}} > \quad i = 1,2; j = 1,2,...,n_i$

Let us define the corresponding parameter uncertainty box

$$Q_i = \{p_i : \underline{p_{ji}} \leq p_{ji} \leq \overline{p_{ji}}, i = 1,2; j = 0,1,2,...,n_i\} \tag{8}$$

The global parameter uncertainty box is then $Q{=}Q_1{\times}Q_2$

The following assumptions about the linear interval polynomials are considered:

- Elements of $p_i \in Q_i, i = 1,2$ are perturbed independently of each other. Equivalently, Q is (n_1+n_2) axis parallel rectangular box.
- Characteristic polynomials of the plant and the controller are of the same degree.

According to [bhata] the closed-loop stability problem can be solved using the *Generalized Kharitonov Theorem.*

Theorem 1

For a given $R(s) = [R_1(s)R_2(s)]$ of real polymials: $R(s)$ stabilizes the linear interval polynomials $P(s) = [P_1(s)P_2(s)]$ for all $p \in Q$ if and only if the controller stabilizes the extremal transfer function

$$G_E(s) = \{\frac{K_1(s)}{S_2(s)} \bigcup \frac{S_1(s)}{K_2(s)}\} \tag{9}$$

Moreover, if the controller polynomials $R_i(s), i = 1,2$ are of the form

$$R_i(s) = s^{t_i}(a_i s + b_i)U_i(s)Z_i(s) \tag{10}$$

then it is sufficient if the controller $R(s)$ stabilizes the *Kharitonov transfer function*

$$G_K(s) = \frac{K_1(s)}{K_2(s)} \tag{11}$$

where $K_i(s) = \{K_i(s)^1, K_i(s)^2, K_i(s)^3, K_i(s)^4\}$ stand for Kharitonov polynomials corresponding to each $P_i(s)$ [khar] and

$$\begin{aligned}S_i(s) = \{[K_i(s)^1, K_i(s)^2], [K_i(s)^1, K_i(s)^3], [K_i(s)^2,\\ K_i(s)^4], [K_i(s)^3, K_i(s)^4]\}\end{aligned} \tag{12}$$

stand for *Kharitonov segments* for corresponding $P_i(s)$; $U_i(s)$ is anti-Hurwitz polynomial; $Z_i(s)$ is an even or odd polynomial; a_i, b_i are positive numbers and $t_i \geq 0$ is an integer. Note that:

$S_i(s)^1 = \lambda K_i(s)^1 + (1-\lambda)K_i(s)^2 \quad \lambda \in < 0 \quad 1 >$.

Let the plant transfer function of $G(s)$ be written in the following affine form

$$G(s) = \frac{P_1(s)}{P_2(s)} = \frac{P_{0,1}(s) + \sum_{i=1}^p P_{i,1}(s)q_i}{P_{0,2}(s) + \sum_{i=1}^p P_{i,2}(s)q_i} \tag{13}$$

where $P_{j,1}(s), P_{j,2}(s)$, $j = 0,1,...,p$ are real polynomials with constant parameters and the uncertainty parameter q_i is from the interval $q_i \in < \underline{q_i}, \overline{q_i} > \quad i = 1,2,...,p$

The description (13) represents a polytope of linear systems with the vertices

$$G_{vj}(s) = \frac{P_{v1,j}(s)}{P_{v2,j}(s)} \quad j = 1,2,...,N; N = 2^p \tag{14}$$

computed for different variables $q_i(s), i = 1,2,...,p$ taking alternatively their maximum $\overline{q_i}$ and minimum values

$\underline{q_i}$. Based on the Edge theorem [barlet] the following results can be obtained.

Theorem 2

The controller $R(s) = [R_1(s)R_2(s)]$ with real polynomials stabilizes the affine system (13) for all $q \in Q$ if and only if the controller stabilizes the following extremal transfer function

$$G_P(s) = \frac{\lambda P_{v1,i} + (1-\lambda)P_{v1,j}}{\lambda P_{v2,i} + (1-\lambda)P_{v2,j}}$$
$$\lambda \in < 0,1 > \quad i \neq j, i,j = 1,2,...,p2^{p-1} \tag{15}$$

Both i and j have to be taken as e vertices numbers of corresponding edges. In general, the sets of extremal transfer functions (9) and (15) are quite different. Whilst the number of $G_E(s)$ is equal to 32, the number of $G_P(s)$ depends exponentially on the number of uncertain parameters q_i

For the case of multilinear uncertainty consider the following uncertain plant transfer function

$$G(s) = \frac{P_{11}(s)P_{12}(s)...P_{1n}(s)}{P_{21}(s)P_{22}(s)...P_{2d}(s)} \tag{16}$$

where $P_{ij}(s), i = 1,2; j = 1,2,...,n(d)$ belong to a linear interval polynomial specified as follows

$$p_k^{i,j} \in < \underline{p_k^{i,j}}, \overline{p_k^{i,j}} >, \quad i = 1,2 \quad j = 1,2,...,n(d), \quad k = 0,1,...,n_{ij}(d_{ij})$$

with independently varying parameters.

Let $K_{ij}(s)$ and $S_{ij}(s)$ denote Kharitonov polynomials [khar] and Kharitonov segments of corresponding $P_{ij}(s)$,respectively. The following theorem holds [bhata].

Theorem 3

The controller $R(s) = [R_1(s)R_2(s)]$ stabilizes the multilinear system (16) for the uncertainty box if and only if the polynomials $R(s)$ stabilizes the following extremal transfer function

$$M_E(s) = \{\frac{S_{11}(s)...S_{1n}(s)}{K_{21}(s)...K_{2d}(s)} \bigcup \frac{K_{11}(s)...K_{1n}(s)}{S_{21}(s)...S_{2d}(s)}\} \tag{17}$$

For the sake of limited space, other uncertainty types will not be considered here. For more detail see [bhata], [grman].

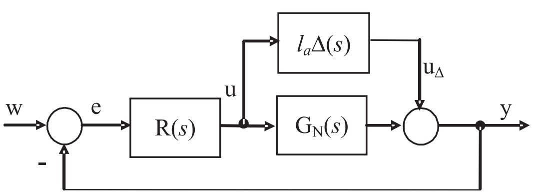

Fig. 2. Standard feedback configuration with additive uncertainty

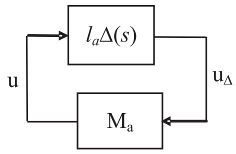

Fig. 3. $M - \Delta$ structure of closed-loop system

For different uncertainty types the following results have been obtained

$$M_a(s) = -[I + R(s)G_N(s)]^{-1}R(s)$$
$$M_i(s) = -[I + R(s)G_N(s)]^{-1}R(s)G_N(s) \quad \forall s \in D$$
$$M_o(s) = -G_N(s)R(s)[I + G_N(s)R(s)]^{-1} \tag{18}$$

Robust stability conditions are given in following theorem [8].

Theorem 4

Assume that the nominal closed-loop system $M_k(s)$, $k = a,i,o$ is stable and the uncertainties satisfy the following inequality

$$0 < l_k(s) \leq l_{km}(s) \quad k = a,i,o \tag{19}$$

q Then the $M - \Delta$ system is stable for all uncertainty models $l_k(s), k = a,i,o$ satisfying (19) if and only if

$$\sigma_M(M_k(s)) < \frac{1}{l_k(s)} \tag{20}$$

The resulting robust decentralized controller design procedure consist of two steps:

- designing a controller $R(s)$ which guarantees stability and performance for nominal plant $G_N(s)$ (nominal stability)
- verification of the robust stability condition (20)

Nominal closed-loop stability under a decentralized controller ($G_N(s)$ and $R(s)$) is guaranteed if and only if the following conditions are satisfied [sko].

Theorem 5

The feedback system in Fig. 1 is stable if and only if

- $\det(F(s)) \neq 0 \quad \forall s \in D$
- $N[0, \det(F(s))] = n_l$ where $F(s) = I + G_N(s)R(s)$ is the return difference matrix, $N[0, \det(F(s))]$ denotes the number of anticlockwise encirclements of the point $[0, 0i]$ by the Nyquist plot of $det(F(s))$, n_l is the number of open-loop unstable poles, *ie*of $G_N(s)R(s)$, and $D = \{s = j\omega : \omega \in (-\infty, \infty)\}$.

Let us factorize $det(F(s))$ in Theorem 5 as follows:

$$det(F(s)) = det(I + G_N(s)R(s))$$
$$= det((R(s)^{-1} + G_d(s) + G_m(s))det(R(s))$$

$$det(F(s)) = det(F_o(s))det(R(s)) \qquad (22)$$

where $G_N(s) = G_d(s) + G_m(s)$, $G_d(s) = diag\{G_N(s)\}$ and $F_o(s) = R(s)^{-1} + G_d(s) + G_m(s)$.

Existence of $R(s)^{-1}$ is implied by the assumption that $det(R(s)) \neq 0$. Using (22), Theorem 5 reads as follows.

Corollary 1

Closed-loop system comprising the nominal model $G_N(s)$ and the controller $R(s)$ is stable if and only if

- $\det(F_o(s)) \neq 0 \quad \forall s \in D$
- $$N[0, \det(F_o(s))] + N[0, \det(R(s))] = n_l \qquad (23)$$

If $R(s)$ has no poles in the open RHP , $N[0, \det(R(s))] = 0$ and the encirclement condition (23) reduces to

$$N[0, \det(F_o(s))] = n_l \qquad (24)$$

Consider a diagonal matrix $P(s)$ such that the following two corollaries hold.

Corollary 2

Let $P(s) = R(s)^{-1} + G_d(s)$. The nominal closed-loop system is stable if either of the first two conditions and the third condition are met

- $det(P(s) + G_m(s)) \neq 0 \quad \forall s \in D$
- $N[0, det(P(s) + G_m(s))] = n_m$ where n_m denotes the number of unstable poles of matrix $P(s) + G_m(s)$
- either of the matrices $M_k(s), k = a, i, o$ is stable.

For example, for $k = a$ we obtain $M_a(s) = -[P(s) + G_m(s)]^{-1} = -\frac{adj[P(s)+G_m(s)]}{det(P(s)+G_m(s))}$ The matrix $M_a(s)$ is stable if and only if the closed-loop characteristic polynomial $p_c(s) = det(P(s) + G_m(s))$ has all roots in the left half complex plane.

Remark 1

Corollary 2 implies

- $det(P(s) + G_m(s)) = det(P(s))det(I + P(s)^{-1}G_m(s))$ Because $P(s)$ is a diagonal matrix numerators of all its entries are to be stable. According to the small gain theorem the necessary and sufficient condition for closed-loop stability (if both transfer function matrices $P(s)^{-1}$ and $G_m(s)$ are stable) reduces to

$$||P(s)^{-1}G_m(s)|| < 1 \Leftrightarrow \sigma_M(G_m(s)) < \sigma_m(P(s)) \qquad (aa)$$

Inequality (aa) has to be fulfilled for all subsystems.

- In the sequel two methods for selecting the diagonal matrix $P(s)$ are presented. For different entries of $P(s) = diag\{P_i(s)\}_{m\times m}$ the following approach can be applied. Due to that matrices P, R, G_d are diagonal the choice of i-th entry $P_i(s)$ of $P(s)$ is following. $P_i(s) = R_i(s)^{-1} + G_{di}(s) \quad i = 1, 2, ..., m$ or $P_i(s) = \frac{P_{ni}(s)}{P_{di}(s)} = \frac{R_{di}G_{ddi}+G_{dni}R_{ni}}{R_{ni}G_{ddi}}$ Denote $P_{ni} = R_{di}G_{ddi} + G_{dni}R_{ni} = R_{ni}\overline{P_{ni}} \quad P_{di} = R_{ni}G_{ddi}$ From above equation one obtains the characteristic polynomial in the form

$$1 + R_i(s)\frac{G_{dni}(s) - \overline{P_{ni}(s)}}{G_{ddi}(s)} = 1 + R_i(s)G^m_{di}(s) \qquad (bb)$$

where the transfer function of i-th modified subsystem is defined as follows

$$G^m_{di}(s) = \frac{G_{dni} - \overline{P_{ni}}}{G_{ddi}} \qquad (aaa)$$

and diagonal transfer function matrix $P(s)$ $P(s) = \{\frac{\overline{P_{ni}(s)}}{G_{ddi}(s)}\}_{m\times m}$ where $\overline{P_{ni}(s)}$ is stable polynomial with corresponding degree such that the conditions of Corollaries 2 and 3 are met.

Denote the following polynomials as follows (index i is omitted) $[p_a(s) = R_dG_{dd} + R_nG_{dn} = a_ns^n + ... + a_0]$ $p_b(s) = R_n\overline{P_n} = b_ms^m + ... + b_0]$ and $p_c(s) = p_a(s) - p_b(s) = c_ks^k + ... + c_0 \quad k = m \quad or \quad k = n$ The following lemma is important for the next development

Lemma 1

We are given two stable polynomials $p_b(s)$ and $p_c(s)$. The polynomial $p_a(s)$ will be stable if one of the following condition is met:

- $\varphi(\omega) = |arg(p_c) - arg(p_b)| < \pi \quad \forall \omega \in \Omega$ where $\Omega = \{\omega : \omega \in < 0 \quad \infty)\}$
- If for some finite number of $\omega_i \quad i = 1, 2, ..., I$ $\varphi(\omega_i) = |arg(p_c) - arg(p_b)| = \pi \quad p_c(\omega_i) \neq -p_b(\omega_i)$ and for $\omega \neq \omega_i$, $\varphi(\omega) < \pi$.

Proof. From Zeros exclusion principle [bhata] the polynomial $p_a(s)$ will be on the boundary of stability if and only if for some $\omega_i \in \Omega$ and stable polynomials $p_c(s)$ and $p_b(s)$ $p_a(\omega_i) = 0 \quad \rightarrow p_c(\omega_i) = -p_b(\omega_i)$ Because for $\varphi(\omega) = 0 \ \forall \omega \in \Omega$ the polynomial $p_a(s)$ is stable, $p_a(s)$ will be stable if $\varphi(\omega) = |arg(p_c) - arg(p_b)| < \pi \quad \forall \omega \in \Omega$ If for some finite number of $\omega_i \quad i = 1, 2, ..., I$ $\varphi(\omega_i) = |arg(p_c) - arg(p_b)| = \pi \quad p_c(\omega_i) \neq -p_b(\omega_i)$ and for $\omega \neq \omega_i$, $\varphi(\omega) < \pi$ the polynomial $p_a(s)$ is stable.

Remark

From the Lemma 1 and the Mikhailov test stability, see for example [mikh] results that for ensure fulfilment of the stability conditions of Lemma 1 the following is recommended:

- $|degree(p_c(s)) - degree(p_b(s))| \leq 2$
- stable roots of $p_c(s)$ should be close to stable parts of roots of G_{dd}.
- the controller transfer function numerator has to be a stable polynomial.

Corollary 3

The closed-loop system in Fig. 3 is robustly stable if for either of the uncertainty types (2),(3) or (4) satisfying (19) and conditions of Corollary 2 with the corresponding below-given inequalities are met:

- for the additive uncertainty

$$\sigma_M([P(s) + G_m(s)]^{-1}) < \frac{1}{|l_a(s)|}$$

- for the input multiplicative uncertainty

$$\sigma_M([P(s)+G_m(s)]^{-1}G_N(s)) < \frac{1}{|l_i(s)|} \qquad (26)$$

- for the output multiplicative uncertainty $\sigma_M(G_N(s)[P(s)\ +G_m(s)]^{-1}) < \frac{1}{|l_o(s)|}$

For identical entries of $P(s) = p(s)I$ the following approach has been developed. From Corollary 2 results

$$I + R(s)[G_d(s) - P(s)] = 0 \qquad (27)$$

which on the subsystem level

$$G_i^{eq} = G_i(s) - p_i(s) \quad i = 1,2,...,m \qquad (28)$$

and controllers $R_i(s)$.

Recall that the characteristic function of $G_m(s)$ are defined as follows $det(g_i(s)I - G_m(s)) = 0 \quad i = 1,2,...,m$ If we consider identical entries in the diagonal matrix $P(s) = p(s)I$, and substitute into the first expression of Corollary 2 and equate it to zero

$$det(p(s)I + G_m(s)) = 0 \qquad (29)$$

we actually obtain a relation for calculating $p(s)$ as a characteristic function of $[-G_m(s)]$
If for a fixed $l \in \{1,2,...,m\}$ $p(s)$ is chosen as $p(s) = -g_l(s)$ then

$$det(F_o(s)) = \prod_{i=1}^{m}[p(s) + g_i(s)] = 0 \qquad (30)$$

In that case according to respect to Corollary 2 the closed-loop system has located poles in the left half plane and some on the imaginary axis. Stability conditions of complex system for the case of identical entries of $P(s)$ are given in the following Theorem [kozv].

Theorem 6

The closed-loop system in Fig. 1 comprising the system $G_N(s)$ and a not unstable decentralized controller $R(s)$ is stable with a degree of stability $\alpha > 0$ if and only if for a selected characteristic function of $-G_m(s-\alpha)$, $p(s-\alpha) = -g_l(s-\alpha)$ there exists a constant α_m such that for all α and any α_1 $0 \le \alpha_1 < \alpha \le \alpha_m$ and $\forall s \in D$ the following conditions hold

$$det(F_o(s)) = \prod_{i=1}^{m}[p(s-\alpha) + g_i(s-\alpha_1)] \neq 0$$
$$\sum_{i=1}^{m} N[0, m_{il}^{eq}(s)] = n_m \qquad (31)$$

where $m_{il}^{eq} = [p(s-\alpha) + g_i(s-\alpha_1)] \quad i = 1,2,...,m$
However, if $\alpha_m \to 0$ and for some $s \in D$ happens that $det(F_o(s)) = \prod_{i=1}^{m}[p(s-\alpha) + g_i(s-\alpha_1)] = 0$ *ie*if the plot of $p(s-\alpha)$ and any characteristic locus $g_i(s-\alpha_1), i = 1,2,...,m$ happen to cross, conditions of Theorem 6 are not met and the closed-loop stability cannot be achieved using the decentralized controller $R(s)$. The above partial results are summarized in the following definition and theorem.

Definition 1

For $l \in \{1,2,...,m\}$ and $\alpha > \alpha_1 \ge 0$ the characteristic function $g_l(s-\alpha)$ of $[-G_m(s-\alpha)]$ is called a stable characteristic function if it satisfies Theorem 6. The set of all stable characteristic functions is denoted P_S.

Theorem 7

The closed-loop system in Fig. 1 comprising the system $G_N(s)$ and a not unstable decentralized controller $R(s)$ is stable with a degree of stability $\alpha > 0$ if and only if :

- $p(s-\alpha) = -g_l(s-\alpha) \in P_S \quad \forall s \in D$ for some fixed $l \in \{1,2,...,m\}$ and $\alpha > \alpha_1 \ge 0$
- all equivalent characteristic polynomials (27) are stable with the roots satisfying $Res \le -\alpha$

EXAMPLES

In the first example the Magnetic levitation model has been considered. The problem is to design a robust PID controller which will guarantee stability and a desired performance in terms of phase margin over the whole operation range of the plant. The magnetic levitation model is described in [magn] and the linearized model is given as follows

$$\dot{x} = Ax + Bu, \quad y = Cx$$

where $x^T = [\Delta x \quad \Delta x_1]$ and

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k_{DA}^2 k_f U_{MUD}^2}{m_k(x_{oo}-x_o)^3} & -\frac{k_{fv}}{m_k} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -\frac{2k_{DA}^2 k_f U_{MUD}}{m_k(x_{oo}-x_o)^2} \end{bmatrix}$$
$$C = [k_{AD}k_x \quad 0]$$

The corresponding transfer function

$$y(s) = C(sI-A)^{-1}Bu(s)$$
$$\to G(s) = \frac{y(s)}{u(s)} = \frac{k_m}{as^2 + bs - 1}$$

For more detail see [magn].

The linear interval model of the magnetic levitation is given as follows

$$k_m \in < 2.4 \quad 6.8 >$$
$$a \in < 1.34 \quad 4.025 > *10^{-4}$$
$$b \in < 1.7975 \quad 5.3895 > *10^{-6}$$

Let the required of closed-loop performance be given in terms of $M_T = 1.6$, $M_S = 2$ and a phase margin more

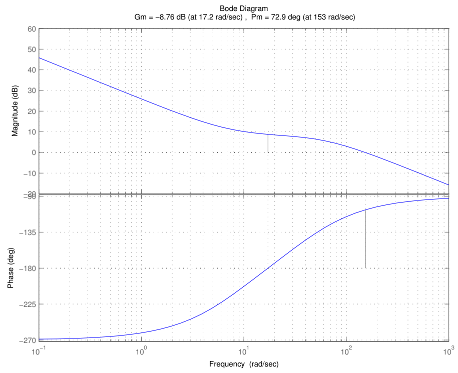

Fig. 4. Bode diagram of open-loop system

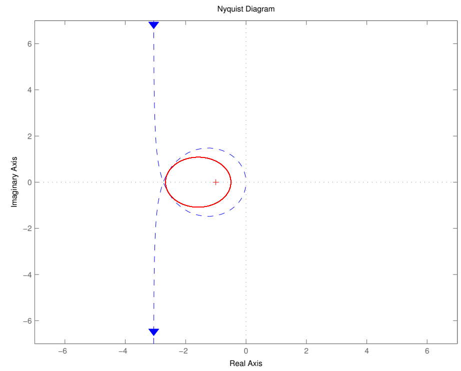

Fig. 5. Nyquist plot with the prohibited area

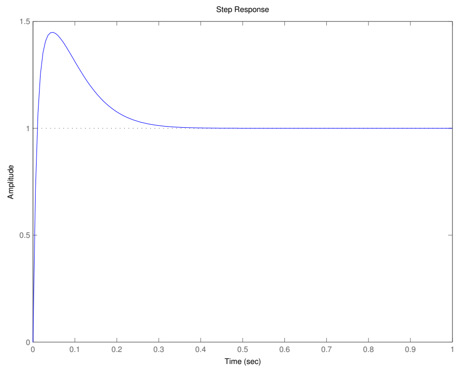

Fig. 6. Closed-loop step response

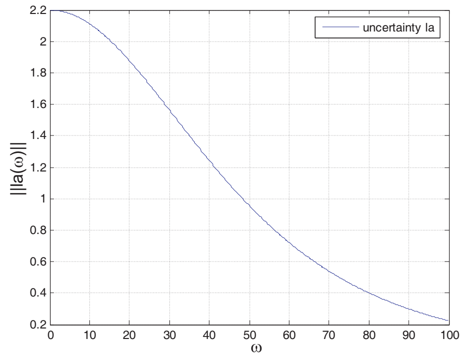

Fig. 7. The additive uncertainty $|l_a(s)|$ versus ω plot 4.5

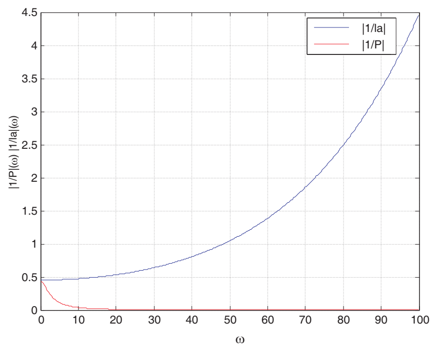

Fig. 8. Verification of the robust stability condition

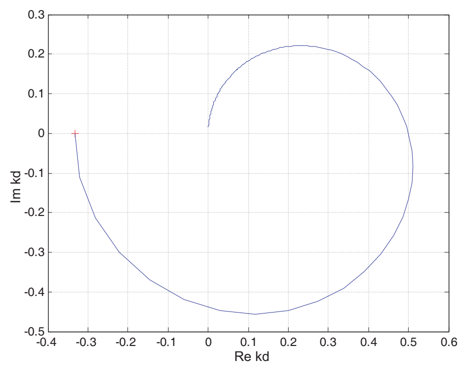

Fig. 9. D-plot for the choice of k_d

than $PM = 72$ degrees. Using the extremal transfer function (9) and the Bode approach the robust PID controller transfer function has been obtained

$$R(s) = \frac{0.02748s^2 + 1.278s + 8.162}{s}$$

The Bode diagram for the worst case open-loop system is in Fig. 4. The Nyquist plot with the circle defining the prohibited area are in Fig. 5

The worst case closed loop step response is given in Fig. 6. Applying *Corollary 2* and *Remark 1* in the robust PID controller design for the above example have obtained the following results. The additive uncertainty $|l_a(s)|$ versus omega plot is depicted in Fig. 7

Taking $\overline{P_n(s)} = (s+60)(s+5)(s+2)$ and applying the D-partition approach the following PID controller is obtained $k_p + k_i/s + k_d s = 1.5 + 3/s + 0.03s$ Verification of the robust stability condition (26) is in Fig. 8; the D-curve for choosing the controller gain k_d is in Fig. 9.

Closed-loop step responses in the two plant working points are in Fig. 10 and Fig. 11, 1-st working point transfer function, Fig. 10: $G_1(s) = \frac{6.8}{0.0004025s^2+5.389^{-6}s-1}$

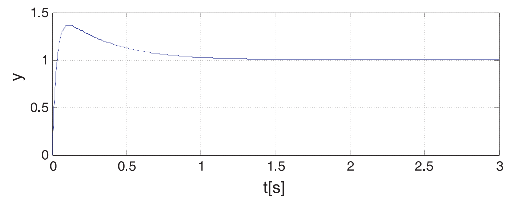

Fig. 10. Closed-loop step response in the 1st working point

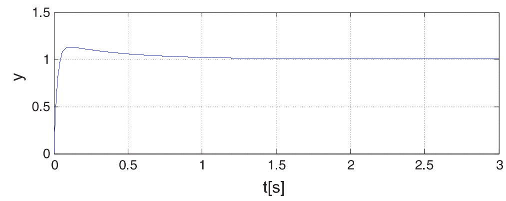

Fig. 11. Closed-loop step response in the 2nd working point

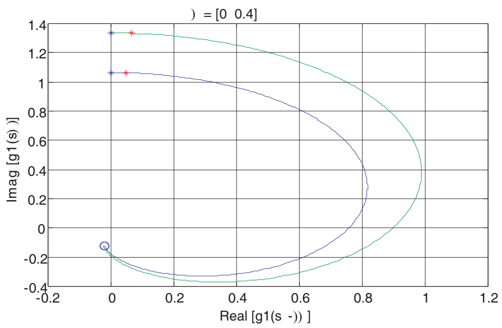

Fig. 12. Characteristic locus $g1(s-\alpha)$ of $G_m(s-\alpha), \alpha = \{0, 0.4\}$

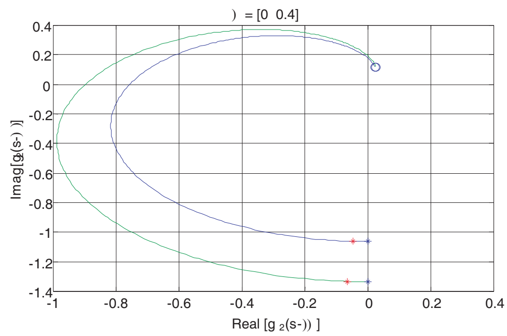

Fig. 13. Characteristic locus $g2(s-\alpha)$ of $G_m(s-\alpha), \alpha = \{0, 0.4\}$

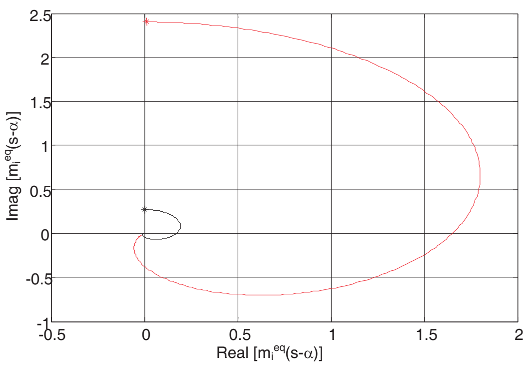

Fig. 14. Equivalent characteristic loci $m_{i1}^{eq}, i = 1, 2$

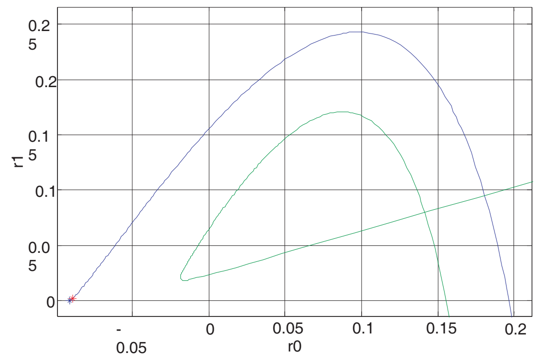

Fig. 15. D-partition of the $(r0, r1)$ plane 1st subsystem

second working point transfer function, Fig. 11: $G_2(s) = \frac{2.4}{0.000134s^2+1.797^{-6}s-1}$

The second example deals with the glass tube drawing plant where the glass metal flowing out from feeder is wrapping around a rotating cylindrical blowpipe. At its lower end, a tube is continuously being drawn using a drawing machine situated at the end of the line. Forming air is blown into the tube under a certain pressure. The produced glass tube has to have required parameters: outer diameter and wall thickness; these quantities are manipulated through the pressure of the forming air and the drawing speed of the drawing machine. Assume pairing of the input and output variables defining individual subsystems to be completed as follows:

u_1- blowing air pressure

u_2 -speed of drawing

y_1 -outside diameter of the tube and

y_2 -tube wall thickness.

The process was linearized in several operating points. The below transfer function matrix corresponds to one chosen operating point.

$$G(s) = \begin{bmatrix} \frac{187e^{-.5s}}{s^2+10.6s+17.2} & \frac{5.45(s-4.5)}{s^2+11.85s+27.95} \\ \frac{25}{s^2+8.84s+19.52} & \frac{57.5}{s^2+13.42s+39.76} \end{bmatrix}$$

The objective is to design two local decentralized PID controllers guaranteeing that the pre-set output parameters (wall thickness, outside tube diameter) are maintained and the whole process is robustly stable within 15 percent of plant parameter changes. The design procedure is as follows: The characteristic loci (CL) of $G_m(s-\alpha)$ for $\alpha = \{0, .4\}$ are plotted in Fig. 12 and Fig. 13. One of them has been chosen to generate P(s).

Consider the first characteristic locus $g1(s-\alpha)$ and specify $p(s)$ to be $p(s) = -g1(s-0.4)$; the corresponding

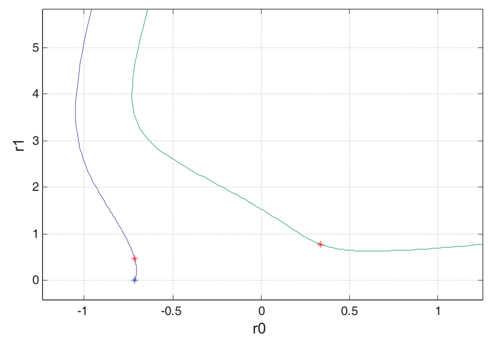

Fig. 16. D-partition of the $(r0, r1)$ plane 2nd subsystem

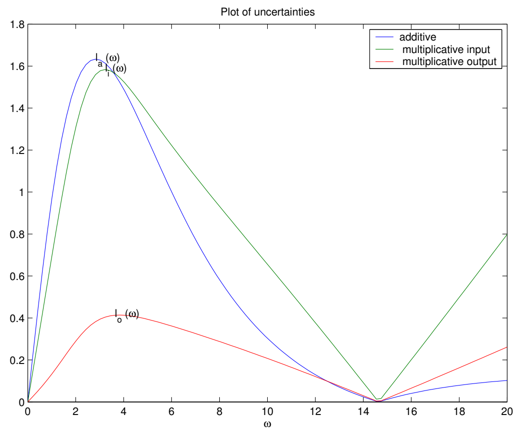

Fig. 17. Plot of three type uncertainties

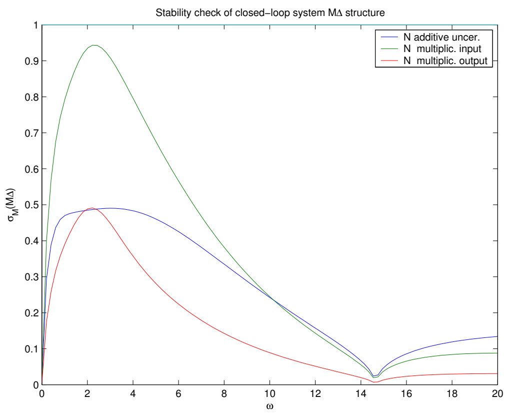

Fig. 18. The robust stability check of closed-loop system

equivalent characteristic loci $m_{i1}^{eq} = [g1(s-.4) - g_i(s), i = 1, 2$ are plotted in Fig. 14.

According to *Definition 1* $g1(s - .4)$ is a stable characteristic locus. Next, the D-partition method has been applied to both equivalent subsystems obtained by modifying the Nyquist plots of decoupled subsystems through the chosen characteristic locus g1(s-.4). Corresponding D-plots in the $(k_p = r0, k_i = r1)$ plane for the first and second subsystems are in Fig. 15 and Fig. 16, respectively .

From the boundary plots of the stable controller parameter regions with degree of stability $\alpha = .4$ the following PI controller parameters have been chosen $R_1(s) = .047 + \frac{.0564}{s}$ $R_2(s) = 0.3999 + \frac{.7041}{s}$ The closed-loop poles are as follows $eigCL = \{-.4232 \pm 0.1632i;$ $-0.8181; -1, 277; -1.3803;$ $-6.4557 \pm 3.55i;$ $-7.3585;$ $-10.7034 \pm 5.1116i;$ $-14.4123\}$ The above designed local PI controllers guarantee stability of the full nominal closed-loop system with the achieved degree of stability $\alpha = 0.4232$. Assume that all parameters of the plant transfer function vary within ± 15 percent around their nominal value;thus the uncertain system can be described by 3 transfer function matrices corresponding to the nominal model, the +15 percent model and the -15 percent model. After evaluating the plant uncertainty using (2), (3), (4) the three plots in Fig. 17 have been obtained. To verify robust closed-loop stability under the decentralized controller designed for the nominal model (20) has been modified to give

$$l_k(s)\sigma_M(M_k(s)) < 1 \quad k = a, i, o$$

Fig. 18 shows the result of the robust stability test: as all plots (either of them one would suffice) lie below 1, the closed-loop system is robustly stable for the 15 percent changes in all plant parameters.

5 CONCLUSION

In this paper a novel design technique is proposed to guarantee a required performance of the full MIMO system by applying the independent design to the equivalent subsystems.

Acknowledgment

The work has been supported by Grant N 1/3841/06 of the Slovak Scientific Grant Agency.

References [1] Ackerman J., Barlett A., Kaesbauer D., Sienel W. and Steinhauser R. : Robust Control: Systems with Uncertain Physical Parameters, Springer-Verlag , 1997.

[2] Barlett A.C., Hollot C.V. and Lin H.: Root Location of an Entire Polytope of Polynomials: It Su?ces to check the edges, Mathematics of Control, Signals and Systems, 1,pp 61-71, 1988.

[3] Bhattacharyya S.P., Chapellat H. and Keel L.H.: Robust Control: The Parametric Approach, Prentice Hall, Englewood Cli?s, 1995 a y V. and Koza

[4] Grman L., Rosinov D., VeselakovA.: Robust Stability Conditions For Polytopic Systems, IJSS Vol 36, N15 15 December, 2005, 961-973.

[5] Kharitonov V.L. : Asymptotic stability of an equilibrium position of a family of systems of linear di?erential equations, Di?erential Equations, 14, 1979, 1483-1485.

[6] Koza y V. :Independent Design of Decentralized ConakovA. and Veseltrollers for Speci?ed Closed-loop Performance, Proc. European Control Conference, Cambridge, Uk , 2003 CD-ROM.

[7] Humusoft: Magnetic Levitation Model, Users manual 1996-2002.

[8] Skogestad S. and Postlethwaite: Multivariable Feedback Control: Analysis and Design, John Wiley and Sons, 1996