

Robust stability conditions for polytopic systems

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Abstract: The paper provides survey of some recent robust stability conditions, their mutual comparison, and presents new robust stability conditions for continuous- and discrete-time systems with convex polytopic uncertainty. Robust stability analysis is based on LMI conditions and parameter-dependent Lyapunov functions. The developed stability conditions are appropriate for output feedback design. Numerical examples thoroughly illustrate power of the considered robust stability analysis methods and show which of them provides the less conservative results.

Keywords: robust stability, Lyapunov function, LMI conditions

1 INTRODUCTION

During the last two decades, robustness has been recognized as a key issue in the analysis and design of control systems. The field of robust control methods based on small-gain-like robustness conditions started developing with the pioneering work of Zames (Zames 1981) where the consequence of the robust control paradigm was definition of the control design problem as an optimization problem. Only at the end of 80s a practical solution to this problem was found in (Doyle *et al.* 1989, Fan *et al.* 1991). It is worth to mention some algebraic approaches, which followed the seminal work of Kharitonov (Kharitonov 1979, Bhattacharyya *et al.* 1995). For convex polytopic uncertainty the Edge theorem (Bartlett *et al.* 1988) and related works provide stability conditions for polytopic systems (Bhattacharyya *et al.* 1995). The recent approach to which belongs this paper, covers the class of Lyapunov-like and LMI methods (Boyd *et al.* 1994).

Description of uncertain systems using the convex polytope-type uncertainty has found its natural framework in the LMI formalism. The LMI stability analysis of such systems is based upon the quadratic stability. A drawback of quadratic stability is that it guards against arbitrary fast parameter variations and thus it uses a single Lyapunov function for testing stability over the whole uncertainty box (Boyd *et al.* 1994). To reduce quadratic stability conservatism in analyzing robust stability of polytopic systems, the affine Lyapunov function has been proposed in (Gahinet *et al.* 1996) and parameter-dependent Lyapunov function (denoted as PDLF) has been introduced for both the continuous-time systems (Peaucelle *et al.* 2000, Ebihara Y. and Hagiwara T. 2002, Henrion *et al.* 2002, Takahashi *et al.* 2002, Veselý 2003) and the discrete-time systems (de Oliveira *et al.* 1999). The affine quadratic stability method (Gahinet *et al.*

1996) is the oldest of these methods employing Lyapunov function that varies depending on uncertainty parameter.

Stability analysis conditions developed so far for a linear polytopic system are based on sufficient stability condition; therefore they still include certain level of conservatism. Another important feature of stability condition is its applicability for a robust controller synthesis (well known quality of this kind is dilation: it means that two matrices of interest do not appear in a product). Hence, the reason for development of new stability analysis methods that motivated also the latter mentioned papers could be summarized as:

- to relax a sufficient robust stability conditions and decrease their conservatism as much as possible,
- to propose stability condition in the form that could be easily modified for robust output feedback controller design.

In control synthesis, it is also important to assess and choose the least conservative method of stability analysis.

This paper tackles the following issues:

1. Comparison of numerical results obtained from the robust stability analysis using various parameter-dependent Lyapunov functions for both continuous-time and discrete-time systems.
2. Development of a new robust stability condition for continuous-time polytopic systems, which provides less conservative results in comparison with several recent parameter-dependent Lyapunov methods (Peaucelle *et al.* 2000, Ebihara Y. and Hagiwara T. 2002, Henrion *et al.* 2002, Takahashi *et al.* 2002, Veselý 2003). The new stability condition is required to be directly applicable for output feedback design. The discrete-time stability condition is proposed as well.

The paper is organized in the following way. In Section 2 the problem formulation and robust stability analysis approaches are recalled: the well-known results on quadratic stability and a survey of several recent affine Lyapunov function and parameter-dependent Lyapunov function methods. Section 3 presents the main result: new developed stability conditions for polytopic systems both for continuous and discrete-time system. Solution of numerical examples and a thorough comparison of obtained results with the existing ones are in Section 4.

A standard notation has been used throughout the paper. A real symmetric positive (negative) definite matrix is denoted as $P > 0$ ($P < 0$). Much of the notation and terminology follows (de Oliveira *et al.* 1999, Henrion *et al.* 2002).

2 PROBLEM FORMULATION AND PRELIMINARIES

Consider the class of uncertain linear systems described as

$$\delta(x) = (A_0 + A_1\Theta_1 + A_2\Theta_2 + \dots + A_p\Theta_p)x = A(\Theta)x \quad (1)$$

where $\delta(\cdot)$ denotes the derivative operator for continuous-time systems or the forward difference operator for discrete-time ones, $x \in R^n$ is the state vector, $A_k \in R^{n \times n}$, $k = 0, 1, 2, \dots, p$ are constant matrices, $\Theta = [\Theta_1 \dots \Theta_p] \in R^p$ is vector of uncertain, possibly time varying parameters, assuming their values and rates in intervals

$$\Theta_j \in \langle \underline{\Theta}_j, \bar{\Theta}_j \rangle, \quad \dot{\Theta}_j \in \langle \underline{r}_j, \bar{r}_j \rangle, \quad j = 1, 2, \dots, p \quad (2)$$

where $\underline{\Theta}_j, \bar{\Theta}_j, \underline{r}_j, \bar{r}_j$ are known lower and upper uncertainty bounds, respectively. The system (1) is referred to as an affine parameter dependent model. Let Γ and Λ denote the sets of vertices of the parameter box and of the parameter variation rate box (2), respectively

$$\begin{aligned} \Gamma &= \{(\gamma_1, \dots, \gamma_p) : \gamma_j = \underline{\Theta}_j \text{ or } \gamma_j = \bar{\Theta}_j\} \\ \Lambda &= \{(\lambda_1, \dots, \lambda_p) : \lambda_j = \underline{r}_j \text{ or } \lambda_j = \bar{r}_j\}. \end{aligned} \quad (3)$$

Let

$$\Theta_m = \left[\frac{\underline{\Theta}_1 + \bar{\Theta}_1}{2}, \dots, \frac{\underline{\Theta}_p + \bar{\Theta}_p}{2} \right] \quad (4)$$

denotes the average of the vector of uncertain parameters.

There are two particular cases of robust stability analysis problem.

1. The uncertain parameter vector $\Theta \in R^p$ is a fixed but unknown element of a given parameter set.
2. The uncertain parameter Θ is a time varying function $\Theta : R \rightarrow R^p$ which belongs to some set defined in R^p . The equation (1) is then to be interpreted in the sense of time variant system.

The first case typically appears in models in which the physical parameters are fixed but only approximately known up to some accuracy. Then, uncertain parameter equation (1) defines a linear time invariant system. In this case parameter dependent Lyapunov function has been introduced. Methods based on quadratic stability can be applied in both cases. To enable comparison of numerical results obtained in this paper for the robust stability analysis, the time invariant model will be considered in the sequel.

The parameter dependent model (1) is a polytope of linear affine systems, which can be described by the list of its vertices

$$\delta(x) = A_{v_i} x, \quad i = 1, 2, \dots, N \quad (5)$$

where $N = 2^p$ is number of polytope vertices.

The polytope vertices are computed for different uncertain parameters Θ_j , alternatively taken at their maximum $\bar{\Theta}_j$ and minimum $\underline{\Theta}_j$ for $j = 1, 2, \dots, p$.

The linear uncertain system (5) belongs to a convex polytopic set defined as

$$\delta(x) = A(\alpha)x \quad (6)$$

whereby

$$S := \left\{ A(\alpha) : A(\alpha) = \sum_{i=1}^N \alpha_i A_{v_i}, \sum_{i=1}^N \alpha_i = 1, \alpha_i \geq 0 \right\}. \quad (7)$$

Note that parameter vector $\alpha = [\alpha_1, \dots, \alpha_N]^T$ is fixed but unknown.

In the following definitions and lemmas several recent results on robust stability analysis are presented. Using the concept of Lyapunov stability it is possible to formulate the following definition.

Definition 1

Uncertain system (6) is robustly stable in the uncertainty box (7) if and only if there exists a matrix $P(\alpha) = P^T(\alpha) > 0$ such that

- a) for a continuous-time system

$$A^T(\alpha)P(\alpha) + P(\alpha)A(\alpha) < 0 \quad (8)$$

- b) for a discrete-time system

$$A^T(\alpha)P(\alpha)A(\alpha) - P(\alpha) < 0 \quad (9)$$

for all α such that $A(\alpha) \in S$.

□

According to (de Oliveira *et al.* 1999) there is no general and systematic way to formally determine $P(\alpha)$ as a function of $A(\alpha)$ and uncertain parameter α . Such a matrix $P(\alpha)$ is called the parameter-dependent Lyapunov matrix and for a particular structure of $P(\alpha)$ the inequalities (8) and (9) define the parameter dependent quadratic stability (PDQS).

A simple way to choose $P(\alpha)$ is to look for a single Lyapunov matrix $P(\alpha) = P$. This case, denoted as quadratic stability is characterized in the following lemma.

Lemma 1 (Boyd *et al.* 1994)

Uncertain system (6) is quadratically stable in the uncertain box (7) if and only if there exists a matrix $P = P^T > 0$ such that

- a) for a continuous-time system

$$A_{v_i}^T P + P A_{v_i} < 0, \quad i = 1, 2, \dots, N \quad (10)$$

- b) for a discrete-time system

$$A_{v_i}^T P A_{v_i} - P < 0, \quad i = 1, 2, \dots, N \quad (11)$$

□

Unfortunately, this approach generally provides quite conservative results. To reduce the conservatism when (1) is affine in Θ and matrices $A_k, k = 0, 1, 2, \dots, p$ are time invariant, an affine Lyapunov function $P(\Theta)$ has been introduced (Gahinet *et al.* 1996)

$$P(\Theta) = P_0 + P_1\Theta_1 + P_2\Theta_2 + \dots + P_p\Theta_p \quad (12)$$

Then, the sufficient conditions for affine quadratic stability are given in the next lemma.

Lemma 2 (Gahinet *et al.* 1996)

Consider the linear system (1) and the parameter-dependent Lyapunov function (12). The continuous-time system (1) is affine quadratically stable if $A(\Theta_m)$ is stable and there exist $(p+1)$ symmetric matrices P_0, P_1, \dots, P_p such that $P(\Theta) > 0$ satisfies

$$L(\gamma, \lambda) = A^T(\gamma)P(\gamma) + P(\gamma)A(\gamma) + \dot{P}(\lambda) - P_0 + \sum_{j=1}^p \Theta_j^2 M_j < 0 \quad (13)$$

for all $(\gamma \times \lambda) \in \Gamma \times \Lambda$ and

$$A_j^T P_j + P_j A_j + M_j \geq 0 \quad j = 1, 2, \dots, p \quad (14)$$

where $M_j = M_j^T \geq 0$ are some positive semidefinite matrices.

□

In this case, the stability is guaranteed also for time-varying system with constrained rate of parameter variation. Note that for case $P_1 = P_2 = \dots = P_p = 0$ the conditions (13) and (14) reduces to quadratic stability condition. Affine quadratic stability encompasses quadratic stability but in general can be more conservative than other parameter-dependent quadratic stability concepts.

As it has been indicated above, the next developments consider time-invariant uncertain systems, therefore parameter variation rate is not included. The following parameter-dependent Lyapunov matrix has been used in (Peaucelle *et al.* 2000, Ebihara Y. and Hagiwara T. 2002, Henrion *et al.* 2002, Takahashi *et al.* 2002, Veselý 2003) and in this paper

$$P(\alpha) = \sum_{i=1}^N P_i \alpha_i \quad (15)$$

which has to be positive definite for all values of α such that $A(\alpha) \in S$ (see (7)).

Lemma 3 (Takahashi *et al.* 2002)

The continuous-time system (6) with the parameter-dependent Lyapunov matrix (15) is PDQS if

$$A_{vi}^T P_i + P_i A_{vi} < -I, \quad P_i > 0, \quad i = 1, 2, \dots, N \quad (16)$$

$$A_{vk}^T P_j + P_j A_{vk} + A_{vj}^T P_k + P_k A_{vj} < \frac{2}{N-1} I, \quad k = 1, 2, \dots, N-1, \quad j = k+1, \dots, N \quad (17)$$

where I denotes identity matrix.

□

Less conservative results can be obtained using the following modification of the Lemma 3.

Lemma 4 (Vesely 2003)

The continuous-time system (6) with the parameter-dependent Lyapunov matrix (15) is PDQS if

$$A_{vi}^T P_i + P_i A_{vi} < -M, \quad P_i > 0, \quad i = 1, 2, \dots, N \quad (18)$$

$$A_{vk}^T P_j + P_j A_{vk} + A_{vj}^T P_k + P_k A_{vj} < \frac{2}{N-1} M, \quad k = 1, 2, \dots, N-1, \quad j = k+1, \dots, N \quad (19)$$

where $M = M^T > 0$ is some positive definite matrix. □

The following robust stability analysis approach for both continuous and discrete-time systems can be found respectively in (Henrion *et al.* 2002, Peaucelle *et al.* 2000).

Lemma 5 (Henrion *et al.* 2002)

The (6) with the parameter-dependent Lyapunov matrix (15) is PDQS if there exist a matrix F and matrices $P_i = P_i^T > 0$ satisfying the LMI

$$\begin{bmatrix} F^T A_{vi} + A_{vi}^T F - aP_i & (A_{vi} + F + b^* P_i)^T \\ A_{vi} + F + b^* P_i & 2I - cP_i \end{bmatrix} > 0 \quad i = 1, 2, \dots, N \quad (20)$$

within a stability region in the complex plane defined as

$$D = \left\{ s \in \mathbb{C} : \begin{bmatrix} 1 \\ s \end{bmatrix}^* \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} < 0 \right\} \quad (21)$$

where the asterisk denotes the transpose conjugate. □

Standard choices for D are the left half-plane ($a = 0, b = 1, c = 0$) or the unit circle ($a = -1, b = 0, c = 1$). Obviously, matrix F in (20) is stable.

Lemma 6 (Peaucelle *et al.* 2000)

The system (6) with the parameter-dependent Lyapunov matrix (15) is PDQS if there exist two matrices E, G and matrices $P_i = P_i^T > 0$ satisfying the LMI

$$\begin{bmatrix} EA_{vi} + A_{vi}^T E^T + aP_i & A_{vi}^T G - E + bP_i \\ G^T A_{vi} - E^T + b^* P_i & -G - G^T + cP_i \end{bmatrix} < 0 \quad i = 1, 2, \dots, N \quad (22)$$

within a stability region D in the complex plane (21). □

A general approach to the dilated characterizations in the continuous-time setting has been proposed in (Ebihara Y. and Hagiwara T. 2002). Its authors pursue the idea of introducing a new auxiliary variable to achieve decoupling between the Lyapunov variables and the controller ones (de Oliveira *et al.* 1999, Henrion *et al.* 2002). These nice and interesting features enable multiobjective and robust control in the face of real polytopic uncertainty using the non-common parameter-dependent Lyapunov function (15).

The robust stability analysis results of (Ebihara Y. and Hagiwara T. 2002) are summarized in Lemma 7.

Lemma 7 (Ebihara Y. and Hagiwara T. 2002)

The continuous-time system (6) with parameter-dependent Lyapunov function (15) is PDQS if there exist $P_i = P_i^T > 0$ and a matrix G such that

$$\begin{bmatrix} P_i + (A_{vi} - 0.5I)^T G + G^T (A_{vi} - 0.5I) & -P_i - (A_{vi} - 0.5I)^T G + G^T \\ -P_i + G - G^T (A_{vi} - 0.5I) & -G - G^T \end{bmatrix} < 0 \quad i = 1, 2, \dots, N \quad (23)$$

□

Following results for discrete-time systems provide LMI formulation of the robust stability condition avoiding the product of A_{vi} and P_i (de Oliveira *et al.* 1999), which enables to use parameter dependent Lyapunov function.

Lemma 8 (de Oliveira *et al.* 1999)

The following conditions are equivalent:

- (i) There exists a symmetric matrix $P > 0$ such that

$$A^T P A - P < 0 \quad (24)$$

- (ii) There exist a symmetric matrix P and a matrix G such that

$$\begin{bmatrix} -P & A^T G^T \\ G A & -G - G^T + P \end{bmatrix} < 0 \quad (25)$$

□

Lemma 9 (de Oliveira *et al.* 1999)

Uncertain discrete-time system (5) with parameter-dependent Lyapunov matrix (15) is robustly stable within the uncertainty box (7) if there exist symmetric positive definite matrices $P_i = P_i^T > 0$ and a matrix G such that

$$\begin{bmatrix} P_i & A_{vi}^T G^T \\ G A_{vi} & G + G^T - P_i \end{bmatrix} > 0 \quad i = 1, 2, \dots, N \quad (26)$$

□

3 NEW ROBUST STABILITY CONDITION FOR POLYTOPIC SYSTEMS

In this section, new robust stability conditions for continuous-time and discrete-time polytopic system (6) with the parameter-dependent Lyapunov function (15) are developed. The main result for continuous-time system is stated in the following theorem.

Theorem 1

The continuous polytopic system (6) with the parameter-dependent Lyapunov function (15) is PDQS if there exist real scalar constants v_{ij} and matrices P_i such that

$$A_{v_i}^T P_i + P_i A_{v_i} < -v_{ii} I, \quad P_i > 0, \quad i = 1, 2, \dots, N \quad (27)$$

$$\frac{1}{2} (A_{v_j}^T P_k + P_k A_{v_j} + A_{v_k}^T P_j + P_j A_{v_k}) < v_{jk} I, \quad j = 1, 2, \dots, N-1, \quad k = j+1, \dots, N \quad (28)$$

where $v_{ii} > 0$, $v_{ij} = v_{ji} \geq 0$ for all $i \neq j$ and

$$V = \begin{bmatrix} -v_{11} & \cdots & v_{1N} \\ \vdots & \ddots & \vdots \\ v_{1N} & \cdots & -v_{NN} \end{bmatrix} \quad (29)$$

is a negative definite matrix.

Proof

For $A(\alpha)$ and $P(\alpha)$ given by (7) and (15) respectively, the necessary and sufficient condition (8) can be rewritten as

$$\left(\sum_{i=1}^N \alpha_i A_{v_i} \right)^T \sum_{i=1}^N \alpha_i P_i + \sum_{i=1}^N \alpha_i P_i \sum_{i=1}^N \alpha_i A_{v_i} < 0 \quad (30)$$

or equivalently

$$\sum_{i=1}^N N_{ii} \alpha_i^2 + \sum_{j=1}^{N-1} \sum_{k=j+1}^N 2\alpha_j \alpha_k N_{jk} < 0 \quad (31)$$

where

$$N_{ii} = A_{v_i}^T P_i + P_i A_{v_i}, \quad i = 1, 2, \dots, N$$

$$N_{jk} = \frac{1}{2} (A_{v_j}^T P_k + P_k A_{v_j} + A_{v_k}^T P_j + P_j A_{v_k}), \quad j = 1, 2, \dots, N-1, \quad k = j+1, \dots, N$$

Applying assumptions (27) and (28) to the left hand side of (30) we obtain

$$\sum_{i=1}^N N_{ii} \alpha_i^2 + \sum_{j=1}^{N-1} \sum_{k=j+1}^N 2\alpha_j \alpha_k N_{jk} < I \left(-\sum_{i=1}^N \alpha_i^2 v_{ii} + \sum_{j=1}^{N-1} \sum_{k=j+1}^N 2\alpha_j \alpha_k v_{jk} \right) = \alpha^T V \alpha \quad (32)$$

where $\alpha^T = [\alpha_1 \dots \alpha_N]$ and V is given by (29).

According to the last assumption in Theorem 1, V is negative definite, therefore

$$\sum_{i=1}^N N_{ii} \alpha_i^2 + \sum_{j=1}^{N-1} \sum_{k=j+1}^N 2\alpha_j \alpha_k N_{jk} < \alpha^T V \alpha < 0 \quad (33)$$

which proves the stability condition (30). Hence, assumptions (27), (28), (29) provide sufficient condition to (8), which completes the proof. \square

In the Lemma 3 and Lemma 4 the upper bounding matrices for all vertices (16) and (18) and edges (17) and (19) are given with the same constant unity matrix or positive definite matrix M , respectively. In the Theorem 1 for all vertices (27) and edges (28) the upper bounding matrices are calculated as different diagonal matrices $v_{ij}I$ for $i, j = 1, 2, \dots, N$. Therefore evidently, Theorem 1 encompasses the results of Lemma 3 and it may also relax the conditions of Lemma 4 in the sense that there exist cases for which stability condition (27)-(29) is satisfied, while (18) and (19) not (see the results of calculations in Section 4).

Note that, if $P_1 = P_2 = \dots = P_N = P$, the parameter dependent quadratic stability conditions of Lemma 3, Lemma 4 and Theorem 1 reduces to quadratic stability conditions.

The results on robust stability of discrete-time system in Section 2 (Lemmas 7 and 8) are efficient for analysis as well as for state feedback controller design. However, in the case that static output feedback is to be designed - the product of A_{vi} and G appearing in Lemma 9 should be avoided so that the respective matrix inequality remains linear.

A new robust stability LMI condition for discrete-time systems that does not include a product of the system matrix with other unknown matrix is formulated in the following theorem.

Theorem 2

Uncertain discrete-time system (6) with the parameter-dependent Lyapunov function (15) is PDQS if for some $D_i = D_i^T$ there exist symmetric positive definite matrices $P_i = P_i^T > 0$ and a matrix Z satisfying LMI

$$\begin{bmatrix} -P_i & A_{vi}^T & 0 \\ A_{vi} & -2D_i/\rho & D_i^T Z/\rho \\ 0 & Z^T D_i/\rho & -Z - Z^T + P_i \end{bmatrix} < 0 \quad i = 1, 2, \dots, N \quad (34)$$

where $\rho > 0$ is a real constant.

Proof

The proof is analogous to (Rosinová and Veselý 2003) where the sufficient stability condition for discrete-time polytopic system with output feedback was provided.

Theorem 2 comprises the results stated in Lemmas 7 and 8, and inequality bounds on P^{-1} . Lemma 8 provides a hint how to avoid the product of matrices X and Y in LMI form of the term $(X^T Y X - Y)$.

The bound on P^{-1} follows from the inequality

$$(D - \rho P^{-1})^T \frac{P}{\rho} (D - \rho P^{-1}) \geq 0 \quad (35)$$

that holds for any matrices $P=P^T>0$, $D=D^T$ and a real scalar $\rho > 0$. From (35) we obtain the upper bound on $(-P^{-1})$ used in the following developments

$$-P^{-1} \leq -\frac{2}{\rho} D + \frac{1}{\rho^2} DPD \quad (36)$$

Now, let us prove the implication (34) \Rightarrow (9). Right-multiplying the inequality (34) by

$$T = \begin{bmatrix} I & 0 & 0 \\ 0 & I & \frac{1}{\rho} D_i^T \end{bmatrix} \text{ and left-multiplying it by } T^T \text{ we obtain}$$

$$\begin{bmatrix} -P_i & A_{vi}^T \\ A_{vi} & -\frac{2}{\rho} D_i + \frac{1}{\rho^2} D_i^T P_i D_i \end{bmatrix} < 0 \quad (37)$$

Joining inequalities (35) and (36) yields $A_{vi}^T P_i A_{vi} - P_i < 0$, which proves (34) \Rightarrow (9) for one index i . Hence, to prove stability of the overall system it is sufficient to find the parameter-dependent Lyapunov function (15) such that (34) holds for $A(\alpha)$, $P(\alpha)$ and some $D(\alpha)$. As (34) is linear with respect to P_i , A_{vi} , D , multiplying (34) by the related scalar α_i for each i and summing through all $i=1, \dots, N$ whereby considering that $\sum_{i=1}^N \alpha_i = 1$, yields

$$\begin{bmatrix} -P(\alpha) & A^T(\alpha) & 0 \\ A(\alpha) & -\frac{2}{\rho} D(\alpha) & \frac{1}{\rho} D^T(\alpha) Z \\ 0 & \frac{1}{\rho} Z^T D(\alpha) & -Z - Z^T + P(\alpha) \end{bmatrix} < 0 \quad (38)$$

for $D(\alpha) = \sum_{i=1}^N \alpha_i D_i$.

Due to previous arguments inequality (38) implies (9) which completes the proof. \square

There still remains open question, since either matrix Z or matrices D_i together with free scalar parameter ρ are to be chosen. We consider Z as unknown matrix to be calculated, D_i and ρ are given. We used in our calculations $\rho = 5$, initial choice of matrices $D_i = \rho P_{i0}^{-1}$ where P_{i0} are the

solutions of Lyapunov equation in the respective vertices, and in the case that no feasible solution to (34) was obtained, the following iterative procedure is applied taking the next value of $D_i = \rho P_i^{-1}$.

Note that for $P_1 = P_2 = \dots = P_N = P$ (34) is equivalent to quadratic stability condition (11) under the assumption $D = \rho P^{-1}$.

Obviously, for the sake of mere analysis the described procedure can be considered as inefficient, however in the case of output feedback design, the proposed condition is directly applicable in the presented form since the system matrix does not appear in any product, therefore output feedback gain matrix can be included into the set of unknown matrices without violating the linearity of matrix inequality (34). This feature distinguishes our condition from other ones listed above. Conditions given in Lemmas 5, 6 or 9 all include product of the system matrix with unknown matrix (F , E and G respectively), therefore could be directly applied for output feedback design only if the respective unknown matrix is selected in some way.

All the above described stability analysis methods provide sufficient stability conditions which naturally suggests the question of their conservatism. The respective qualities of the individual methods are illustrated in the next section.

4 EXAMPLES

4.1 Method of evaluation

In this section the properties and power of individual methods presented in Sections 2 and 3 have been tested on several benchmark examples as well as on 1000 randomly generated ones. To be able to evaluate the conservatism of each particular method, the term ‘stability region size’ has been adopted. In each tested example, it has been measured in terms of the parameter q corresponding to the maximum uncertainty polytope for which the uncertain system still remains stable.

The motivation for the adopted approach corresponds to the robust control design task in the following interpretation.

Consider an uncertain affine linear system

$$\delta(x) = A(\Theta)x + B(\Theta)u \quad (39)$$

where $u \in R^m$ is the input vector

$$B(\Theta) = B_0 + B_1\Theta_1 + \dots + B_p\Theta_p \in R^{n \times m}. \quad (40)$$

Consider the static output feedback

$$u = Ky = KCx \quad (41)$$

where $y \in R^l$, $C \in R^{l \times n}$ are output variable and output matrix of the linear system (39), respectively.

The corresponding closed loop system is then

$$\delta(x) = (A(\Theta) + B(\Theta)KC)x = A_c(\Theta)x \quad (42)$$

In the sequel we assume that the affine model is recalculated so that all uncertainties in (42) are non-dimensional and normalized so that

$$q = |\underline{\Theta}_1| = |\overline{\Theta}_1| = \dots = |\underline{\Theta}_p| = |\overline{\Theta}_p|, \quad (43)$$

A_k in (1) and B_k in (40), $k = 0, 1, 2, \dots, p$ are constant.

The matrices A_k and B_k , $k = 0, 1, 2, \dots, p$ were created considering $q = 1$, that is $\underline{\Theta}_i = -1$ and $\overline{\Theta}_i = 1$, $i = 1, 2, \dots, p$. In the assessment of the considered stability conditions the polytope vertices are computed for different q , but the matrices A_k and B_k remain fixed. It is important to note that uncertainties are normalized into the same scale $(-1, 1)$ and scalar parameter q is without physical dimension to enable better assessment of the considered methods.

In this case, the closed loop robust stability analysis problem can be extended as a question may arise about ‘how robust’ the closed loop with the considered controller is:

What is the maximum range for uncertainty parameter q such that the closed loop affine uncertain system (42) remains stable?

This section provides numerical examples, the results have been tested, and thoroughly evaluated and compared with respect to the eight continuous-time and four discrete-time robust stability conditions presented in Sections 2 and 3.

The following test has been applied: first, all considered methods were tested on four continuous-time and one discrete-time models of real plants taken from references; then for the continuous-time case, 1000 affine stable closed loop systems (42) were generated considering $q = 1$ and the pairs $(n = 3, p = 2)$, $(n = 5, p = 2)$ and $(n = 5, p = 3)$; finally, in each example, the maximum value of the uncertainty parameter q was evaluated for each considered robust stability condition, while the matrices A_k and B_k , $k = 0, 1, 2, \dots, p$ are constant. Similarly, for the discrete-time case, 1000 affine stable closed loop systems (42) were generated.

The obtained results have been evaluated as follows:

1. For each example, all methods were arranged according to the maximum value of the uncertainty parameter q ($q_{\max} = \max(q)$) and number of points assigned with respect to their rating (the highest value of q_{\max} - best rating = 1 point, ... etc.), i.e. the fewer points, the better rating of the respective method.
2. For each method, the mean value q_m of all maximum uncertainty parameters q_{\max} obtained in the considered examples was computed ($q_m = (\sum_{r=1}^{1000} q_{\max})/1000$), and the methods were arranged according to decreasing values of q_m . Hence, in this case, the higher value of q_m , the better rating of the respective method.
3. For each generated continuous-time and discrete-time example and for all methods, the percentage of appearance of each point (assigned with respect to their rating) was

computed. This criteria illustrates the relative success of particular method in more details – the whole scale of point distribution is shown.

These three proposed criteria have been chosen due to their obvious interpretation. While the first criterion evaluates the method's rating ('the fewer points – the better method'), the second one estimates the 'size' of the stability region ('the higher q_m – the better method') and the third one evaluates percentage of appearance of each point assigned according to the maximum value of the uncertainty parameter q .

Note, that if a particular method achieves the best rating it is assigned one point, if two methods both achieve the best rating, both of them are assigned one point, however, the next method is assigned 3 points, etc.

4.2 Results for real plants

Example 1

Consider a decentralized MIMO PI controller to be designed for the linear continuous-time model of a laboratory plant comprising two cooperating DC motors. The plant model is given by (39) and (41) with $p = 2$, $n = 10$. The entries of matrices (39) and of the gain matrix K were taken from (Vesely 2002: Example 4)

$$K = \begin{bmatrix} -1.2922 & 0 & -0.2346 & 0 \\ 0 & -2.1924 & 0 & -0.4227 \end{bmatrix}.$$

The robust stability analysis results of are summarized in Table 1.

Table 1

Method	AQ	VES	PEAU	EBI	HEN	MTAKA	Q	TAKA
Rating	1	2	2	4	5	6	7	8
q_{\max}	2.4625	2.4609	2.4609	2.4305	2.3484	2.1852	2.0289	1.1844

where the above acronyms have the following meaning

- AQ – affine quadratic stability (Gahinet *et al.* 1996), *Lemma 2*
- VES – Parameter dependent quadratic stability proposed in this paper, *Theorem 1*
- PEAU – Parameter dependent quadratic stability (Peaucelle *et al.* 2000), *Lemma 6*
- EBI – Parameter dependent quadratic stability (Ebihara Y. - Hagiwara T. 2002), *Lemma 7*
- HEN – Parameter dependent quadratic stability (Henrion *et al.* 2002), *Lemma 5*
- MTAKA – Parameter dependent quadratic stability modified in this paper, *Lemma 4*
- Q – quadratic stability (Boyd *et al.* 1994), *Lemma 1 (a)*
- TAKA – Parameter dependent quadratic stability (Takahashi *et al.* 2002), *Lemma 3*
- q_{\max} – maximum value of the uncertainty parameter q

Example 2

Model of this plant has been borrowed from (Benton *et al.* 1999), whereby $p = 2$, $n = 4$ and the gain matrices are

$$K_1 = \begin{bmatrix} 0.8107 \\ 3.7664 \end{bmatrix} \quad K_2 = \begin{bmatrix} -1.538 \\ 2.8187 \end{bmatrix}$$

The robust stability analysis results for K_1, K_2 are in Table 2.

Table 2

Gain	Method	AQ	VES	PEAU	EBI	HEN	MTAKA	TAKA	Q
K_1	Rating	1	1	1	1	1	6	7	8
	q_{\max}	1.7789	1.7789	1.7789	1.7789	1.7789	1.7578	1.5688	1.1844
K_2	Rating	1	1	1	1	7	5	6	8
	q_{\max}	14.073	14.073	14.073	14.073	10.656	14.031	13.668	6.5719

Example 3

The real plant model has been borrowed from (Vesely 2002: Example 2) with $p = 1, n = 3$ and the gain matrix $K = 1.949$. The robust stability analysis results are in Table 3.

Table 3

Method	AQ	VES	PEAU	EBI	MTAKA	HEN	TAKA	Q
Rating	1	1	1	4	5	6	7	8
q_{\max}	7.3609	7.3609	7.3609	7.2242	7.2188	6.8430	6.2273	4.8828

Example 4

The real plant model has been borrowed from (Takahashi 2002) with $p = 1, n = 3$ and the gain matrix

$$K = \begin{bmatrix} -2.4718 & -0.6109 & -3.009 \end{bmatrix}.$$

The robust stability analysis results are in Table 4.

Table 4

Method	AQ	VES	PEAU	MTAKA	TAKA	HEN	EBI	Q
Rating	1	1	1	4	5	6	7	8
q_{\max}	1.0469	1.0469	1.0469	1.0234	0.8023	0.7898	0.6461	0.3305

Example 5 (discrete-time system)

For the discrete-time case, the real plant model has been taken from (Rosinová and Vesely 2002) with $p = 2, n = 3$ and the gain matrix

$$K = \begin{bmatrix} -2.3276 & -8.9563 \\ -0.8589 & 2.7738 \end{bmatrix}$$

The robust stability analysis results are in Table 5.

Table 5

Method	OLI	HEND	QD	D-V
Rating	1	1	1	4
q_{\max}	1.4234	1.4234	1.4234	1.4227

where the above acronyms have the following meaning

OLI – Parameter dependent quadratic stability proposed in (Oliveira *et al.* 1999), *Lemma 9*

HEND – Parameter dependent quadratic stability proposed in (Henrion *et al.* 2002), *Lemma 5*

QD – quadratic stability (Boyd *et al.* 1994), *Lemma 1 (b)*

D-V – Parameter dependent quadratic stability proposed in this paper, *Theorem 2*

q_{\max} – maximum value of the uncertainty parameter q

4.3 Results for generated examples

For a continuous-time case, 1000 stable closed loop affine systems were generated for the pairs $(n = 3, p = 2)$, $(n = 5, p = 2)$ and $(n = 5, p = 3)$. The results have been evaluated as explained in Section 4.1.

Results obtained for the above-considered pairs are summarized in Table 6, Table 7 and the corresponding charts are in Fig. 1, Fig. 2 and Fig. 3.

Table 6

p	n		AQ	VES	PEAU	MTAKA	HEN	EBI	TAKA	Q
2	3	Rating	1.05	1.14	1.20	2.63	2.47	3.75	6.62	6.08
		q_m	2.96	2.92	2.92	2.91	2.84	2.61	2.89	2.25
		σ_s	1.65	1.60	1.60	1.60	1.53	1.22	1.60	1.31
	5	Rating	1.07	1.15	1.24	2.71	2.68	4.56	6.34	7.13
		q_m	1.98	1.96	1.96	1.96	1.89	1.73	1.92	1.33
		σ_s	0.88	0.86	0.86	0.86	0.80	0.60	0.86	0.70
3	5	Rating	1.04	1.22	1.38	3.27	3.21	5.08	6.11	7.39
		q_m	1.62	1.60	1.60	1.60	1.55	1.43	1.58	1.09
		σ_s	0.57	0.56	0.56	0.56	0.52	0.38	0.56	0.46

where rating, q_m (mean value of maximum uncertainty parameter q_{\max}) and standard deviation σ_s have been calculated from the 1000 robust stability analysis assessments.

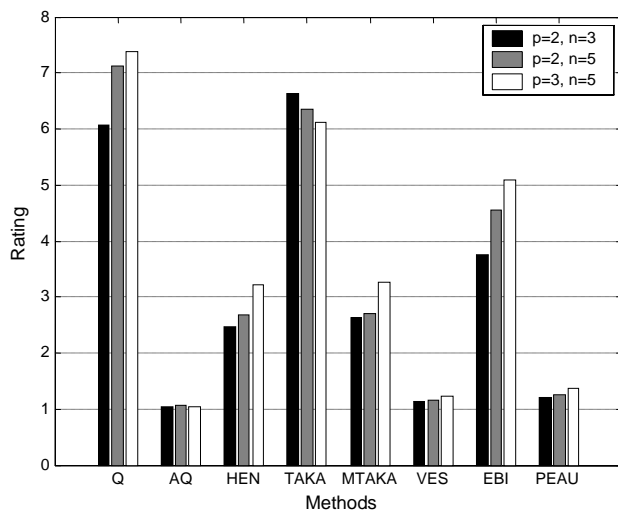


Fig. 1 – Result of robust stability evaluation in terms of rating ('the lower value – the better method')

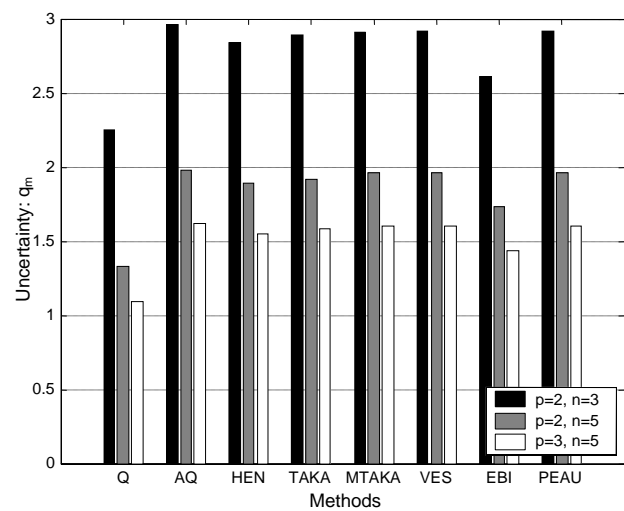


Fig. 2 – Result of robust stability evaluation in terms of q_m ('the higher value – the better method')

Table 7 – Average percentage of appearance of each point for all generated examples

Points	AQ	VES	PEAU	MTAKA	HEN	EBI	TAKA	Q
1	98.10	85.47	83.53	46.03	60.57	39.93	0.067	14.03
2	0.300	12.43	11.27	6.600	4.133	1.033	0	0.767
3	0.400	1.833	1.100	3.733	0.067	0.100	0.267	0
4	0.600	0.167	2.200	17.53	1.300	0.933	2.267	0.033
5	0.500	0.067	1.900	13.43	6.267	3.933	19.47	0.033
6	0.033	0.033	0	9.667	20.47	7.967	30.10	0.533
7	0.067	0	0	3.000	7.200	35.80	34.63	9.233
8	0	0	0	0	0	10.30	13.20	75.37

Percentage of appearance of each point have been calculated from the 3000 robust stability analysis assessments.

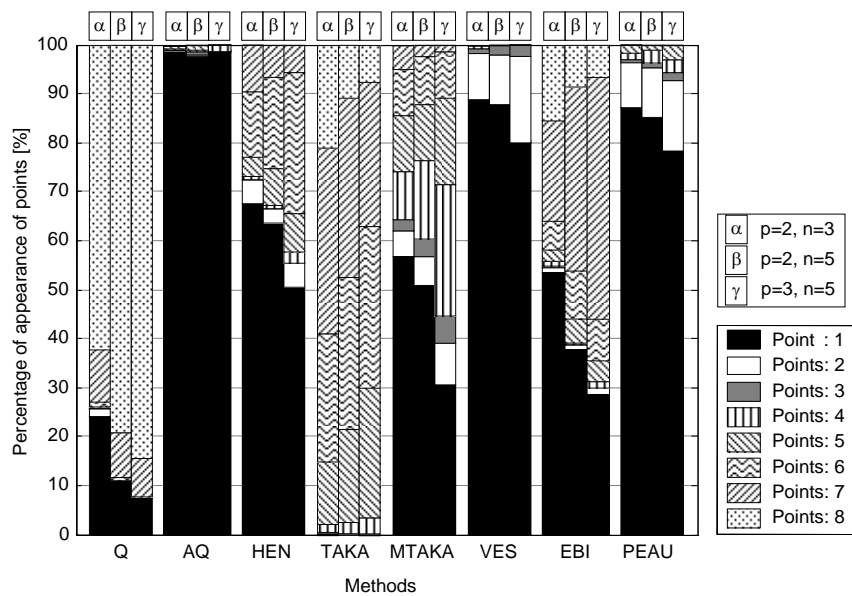


Fig. 3 – Percentage of appearance of points assigned to presented methods

For continuous-time systems, the affine quadratic criterion is undoubtedly the less conservative for all three pairs: AQ criterion has obtained a minimum of points, i.e. AQ evaluation of 3000 affine systems achieves the best rating in 98.1%. The second place goes to the criterion proposed in this paper, taking the best rating in 85.5%. The VES and PEAU criteria provide similar results. The Q criterion has achieved only the two best and the two worst rating positions. When the affine system has become more complex (increased n and p) the quadratic stability criterion has moved to a lower place, and the rating has changed from 6.08 to 7.39. On the contrary, the TAKA criterion's rating has changed from 6.62 to 6.11 and almost does not achieve the three best rating positions. MTAKA, HEN, EBI and PEAU criteria have become more conservative when the affine system has got more complex. It is noteworthy that for all cases, the HEN and MTAKA criteria provide similar results and are less conservative, than the EBI criterion. From a more detailed investigation of Table 6 it is evident that for the TAKA criterion the mean value q_m is larger than for the HEN and EBI criteria, even though according to the

rating, the two latter methods are superior. This contradiction is due to the different value of the standard deviation σ_s . For the third case ($p = 3, n = 5$), the standard deviation values σ_s are given in Table 6 showing that for some examples, the TAKA criterion may provide larger values of q or less conservative results than the HEN and EBI criteria. On the average, for 3000 generated affine systems the TAKA criterion is more conservative than the other six criteria based on the parameter-dependent Lyapunov function.

For the discrete-time case, 1000 stable closed loop affine systems were generated for the pairs ($n = 3, p = 2$), ($n = 5, p = 2$) and ($n = 5, p = 3$). Results are summarized in Table 8, Table 9 and the corresponding charts are in Fig. 4, Fig. 5 and Fig. 6.

Table 8

p	n		QD	HEND	OLI	D-V
2	3	Rating	3.66	2.14	1.46	2.25
		q_m	0.92	1.18	1.30	1.28
		σ_s	0.35	0.45	0.37	0.37
	5	Rating	3.46	3.08	1.29	1.89
		q_m	0.74	0.67	1.20	1.14
		σ_s	0.28	0.58	0.31	0.30
3	5	Rating	3.97	1.46	1.72	2.53
		q_m	0.78	1.21	1.20	1.18
		σ_s	0.23	0.26	0.25	0.24

Rating, standard deviation σ_s and q_m have analogical meaning as in the continuous-time case (Tab.8).

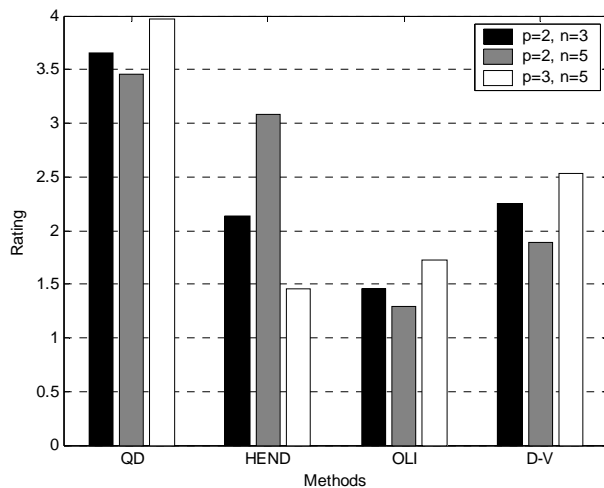


Fig. 4 – Result of robust stability evaluation in terms of rating ('the lower value – the better method')

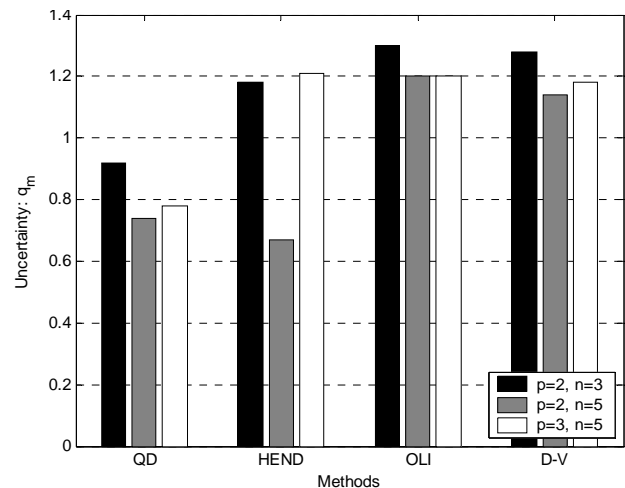


Fig. 5 – Result of robust stability evaluation in terms of q_m ('the higher value – the better method')

Table 9 – Average percentage of appearance of each point for all generated examples

Points	QD	HEND	OLI	D-V
1	2.767	45.17	62.87	17.23
2	0.800	8.033	25.33	45.03
3	20.33	25.90	11.80	35.83
4	76.10	20.90	0	1.900

Percentage of appearance of each point have been calculated from the 3000 robust stability analysis assessments.

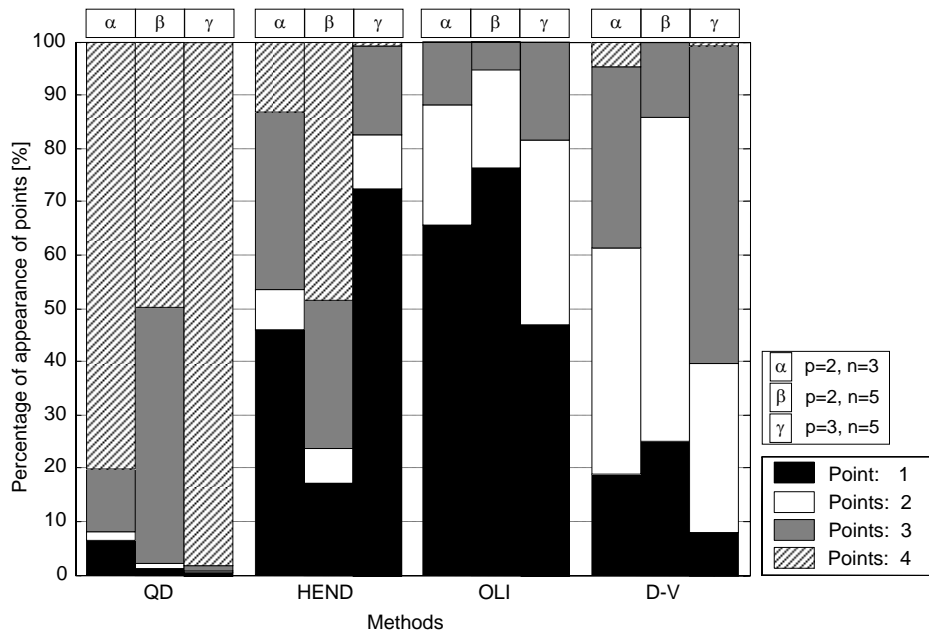


Fig. 6 – Percentage of appearance of points assigned to presented methods

For discrete-time systems, all three parameter-dependent methods (OLI, HEND, D-V) surpass the quadratic stability criterion as expected. The OLI criterion provides the best results in most cases besides the γ case ($p = 3, n = 5$), where HEND is better, as shown in Fig.6. Results obtained by HEND and D-V criteria vary from case to case (for $p = 2, n = 5$, D-V is better than HEND, in the remaining two cases HEND provides better results than D-V). Notice in Tab.9 that D-V method appears on the ‘worst’ position (4 points rating) only in 1.9% of generated examples while HEND in 20.9%. Therefore there exists quite significant set of examples where D-V surpasses HEND. It should be also noted that the D-V criterion proposed in this paper can be is appropriate for the output feedback design, while the OLI criterion can be directly applied just for the state feedback design. Therefore it is believed that the obtained results illustrate the ability of the D-V criterion to successfully compete with the other ones.

5. CONCLUSIONS

The paper provides a numerical comparison of several robust stability analysis methods based on LMI conditions and parameter-dependent Lyapunov functions for both continuous and discrete-

time systems, and proposes new robust stability conditions for continuous and discrete-time polytopic systems. The developed stability conditions are appropriate for output feedback design. According to our comparative test, the robust stability criterion developed for continuous-time systems provides less conservative results when compared with several recent parameter-dependent Lyapunov methods. Obtained results prove that the affine quadratic stability criterion is the least conservative for all considered examples and cases. However, the affine quadratic stability criterion is more complex to be applied for static output feedback design in comparison with other parameter dependent Lyapunov function methods.

The D-V stability criterion proposed for discrete-time systems is based on dilation of the LMI characterization and can be directly used for the output feedback control design (Rosinová and Veselý 2003). The comparison of the considered discrete-time stability criteria favours the OLI criterion, which, however, is not appropriate for output feedback design. The D-V provides results, which in some cases surpass those of HEND and is believed to contribute to the family of LMI stability results for polytopic discrete-time systems.

Thorough analysis of several recent robust stability criteria is given that can help to assess and choose the least conservative method of stability analysis for output feedback controller design.

Acknowledgement

This work has been supported by the Scientific Grant Agency of the Ministry of Education of the Slovak Republic and the Slovak Academy of Sciences under Grant No. 1/0158/03. The authors would like to thank the anonymous reviewers for their valuable comments and suggestions, which improved this paper.

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