EXTREMAL TRANSFER FUNCTIONS IN ROBUST CONTROL SYSTEM DESIGN *

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Abstract: The paper addresses the design of a robust output feedback controller for SISO systems using extremal transfer functions and the classical control theory approach. It focuses on robust stabilization of uncertain plants, which belong to linear or multilinear uncertain systems. A survey on extremal transfer functions is given and it is shown that for the proposed robust controller design procedure the classical linear control theory can be applied providing necessary and sufficient or sufficient stability conditions.

Keywords: robust control, extremal transfer function, interval control system and multilinear interval system

1. INTRODUCTION

Robustness has been recognized as a key issue in the analysis and design of control systems for the last two decades.

The main criticism formulated by control engineers against modern robust analysis and design methods for linear systems concerns the lack of efficient, easy-to-use and systematic numerical tools. Indeed, a lot of analysis techniques and most of the design techniques for uncertain systems boil down to non-convex bilinear matrix inequality (BMI) problems, for which no polynomial-time algorithm has been proposed so far (Henrion, *et al.*, 2002). This is especially true when analysing robust stability or designing robust controllers for MIMO systems affected by highly structured uncertainties, or when seeking a low order or a given order robust controller.*

In this paper we focus on the problem of robust stabilization of an uncertain single input – single output plant, which belongs to linear or multilinear interval systems. A survey on extremal transfer functions is given and it is shown that for the robust controller design procedure the classical linear control theory can be applied with necessary and sufficient or sufficient stability conditions.

Even though significant progress has been made recently in the field of analysing robust stability for parametric uncertain systems, the robust controller design procedure is still an open problem. Indeed, in (Bhattacharyya, *et al.*, 1995) it is pointed out that a significant deficiency of the control theory is lack of no conservative robust controller design methods. Recent developments in the robust control of systems with parametric uncertainty have been inspired by the Kharitonov Theorem (Kharitonov, 1979). By means of this theorem it is sufficient to determine stability only of four Kharitonov polynomials. Kharitonov's theorem has been generalized for the control problem (Chapellat, Bhattacharyya, 1989). The Generalized Kharitonov Theorem shows that for a compensator to robustly stabilize the system it is sufficient if it stabilizes a prescribed set of line segments in the plant parameter space. Under special conditions on compensator it is sufficient to stabilize the Kharitonov vertices. Based on Hermite-Fujiwara matrices and the Generalized Kharitonov Theorem a sufficient condition for the extreme of a robustly stabilizing controller of order up to three is derived in (Henrion *et al.*, 2001). The next important substantial

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progress in the robust analysis of parameter stability is the Edge Theorem (Bartlett, *et al.*, 1988). The Edge Theorem allows to constructively determine the root space of a family of linearly parametrized systems. There are situations when several linear interval systems are connected in series. In such a case the global control object is considered to belong to a multilinear systems class (Chapellat *et al.*, 1994). The main tool to approach this problem is the Mapping Theorem (Hollot, Xu, 1989) which shows that the image set of multilinear interval polynomials is contained in the convex hull of the vertices.

2. PROBLEM STATEMENT

Consider the transfer function

$$G(s) = \frac{P_1(s)}{P_2(s)} \tag{1}$$

where $P_1(s)$, $P_2(s)$ are linear or multilinear interval polynomials with respect to parametric uncertainty described in the parameter box uncertainty Q.

The problem studied in this paper can be formulated as follows: For a continuous time system described by the transfer function (1) the robust controller

$$R(s) = \frac{F_1(s)}{F_2(s)} \tag{2}$$

is to be designed with fixed polynomials $F_1(s)$, $F_2(s)$ such that for the closed loop system

$$G_c(s) = \frac{P_1(s)F_1(s)}{P_1(s)F_1(s) + P_2(s)F_2(s)} \tag{3}$$

robust stability (RS) and a specified robust performance (RP) are guaranteed.

3. EXTREMAL TRANSFER FUNCTIONS

3.1 Linear interval case

We will deal with characteristic polynomials of the form

$$A(s) = F_1(s)P_1(s) + F_2(s)P_2(s) \tag{4}$$

where

$$P_i(s) = p_{0,i} + p_{1,i}s + \cdots + p_{n_i,i}s^{n_i}, \; i = 1,2 \tag{5}$$

Each $P_i(s)$ is a linear interval polynomial specified by the intervals

$$p_{j,i} \in \left\langle \underline{p}_{j,i}, \overline{p}_{j,i} \right\rangle, \; i = 1,2, \; j = 0,1,\ldots,n_i \tag{6}$$

The corresponding parameter box is then

$$Q_i = \left\{ p_i : \underline{p}_{j,i} < p_{j,i} < \overline{p}_{j,i}, \; i = 1,2; \; j = 0,1,\ldots,n_i \right\} \tag{7}$$

and the global parameter uncertainty box is given

$$Q = Q_1 \times Q_2 \tag{8}$$

where $p_i^T = \left[p_{0,i} \; p_{1,i} \cdots p_{n_i,i} \right]$, $i = 1,2$.

Following assumptions about the linear interval polynomials are considered:

1) Elements of $p_i, i = 1,2$ perturb independently of each other. Equivalently, Q is a $n_1 + n_2$ axis parallel rectangular box.
2) All characteristic polynomials (4) are of the same degree.

According to (Bhattacharyya, *et al.*, 1995) the stability problem of (4) can be solved using the Generalized Kharitonov Theorem.

Theorem 1. (Chapellat, Bhattacharyya, 1989)
For a given $F(s) = (F_1(s), F_2(s))$ of real polynomials:

1) $F(s)$ stabilizes the linear interval polynomials $P(s) = (P_1(s), P_2(s))$ for all $p \in Q$ if and only if the controller stabilizes the extremal transfer function

$$G_E(s) = \left\{ \frac{K_1(s)}{S_2(s)} \cup \frac{S_1(s)}{K_2(s)} \right\} \tag{9}$$

2) Moreover, if the polynomials of the controller $F_i(s)$, $i = 1,2$ are of the form

$$F_i(s) = s^{t_i}(a_i s + b_i)U_i(s)Z_i(s) \tag{10}$$

then it is sufficient that the controller F(s) stabilizes the Kharitonov transfer function

$$G_K(s) = \frac{K_1(s)}{K_2(s)} \tag{11}$$

3) Finally, stabilizing (11) is not sufficient to stabilize $P(s) = (P_1(s), P_2(s))$ when the controller polynomials F_i(s), $i = 1,2$, do not satisfy the condition (10)

□

where

$$K_i(s) = \left\{ K_i^1(s), K_i^2(s), K_i^3(s), K_i^4(s) \right\} \tag{12}$$

stand for Kharitonov polynomials (Kharitonov, 1979) corresponding to each $P_i(s)$

$$S_i(s) = \left\{ \left[K_i^1, K_i^2\right], \left[K_i^1, K_i^3\right], \left[K_i^2, K_i^4\right], \left[K_i^3, K_i^4\right] \right\} \tag{13}$$

are the four Kharitonov segments for corresponding $P_i(s)$; U_i(s) is an anti-Hurwitz polynomial, $Z_i(s)$ are even or odd polynomials, a_i, b_i are real positive numbers and $t_i \geq 0$.

Note: $S_i^1 = \lambda K_i^1(s) + (1 - \lambda)K_i^2(s)$, $\lambda \in \langle 0, 1 \rangle$

3.2 Linear affine case

Let the transfer function (1) can be rewritten in the following affine form

$$G(s)=\frac{P_1(s)}{P_2(s)}=\frac{P_{0,1}(s)+\sum_{i=1}^{p}P_{i,1}(s)q_i}{P_{0,2}(s)+\sum_{i=1}^{p}P_{i,2}(s)q_i} \tag{14}$$

where $P_{j,1}(s), P_{j,2}(s)$ for $j = 0, 1, 2, ..., p$ are real polynomials with constant parameters and the uncertain parameter q_i belongs to the interval $q_i \in \langle \underline{q_i}, \overline{q_i} \rangle$, $i = 1, 2, ..., p$.

The system represented by (14) is a polytope of linear systems, which can be described by a list of its vertices

$$G_{Vj}(s)=\frac{P_{V1,j}(s)}{P_{V2,j}(s)},\ j=1,2,\ldots,N,\ N=2^p \tag{15}$$

computed for different permutations of the p variable q_i, $i = 1, 2, ..., p$ alternatively taken at their maximum $\overline{q_i}$ and minimum $\underline{q_i}$.

The characteristic polynomial of polytopic system with controller (2) is given as follows

$$A_{Vj}(s)=F_1(s)P_{V1,j}(s)+F_2(s)P_{V2,j}(s),\ j=1,2,\ldots,N \tag{16}$$

A polytopic family of characteristic polynomials can be represented as the convex hull

$$\Delta(s)=\sum_{j=1}^{N}\lambda_j A_{Vj},\ \lambda_j\geq 0,\ \sum_{j=1}^{N}\lambda_j=1 \tag{17}$$

We make the assumption that all polynomials have the same degree.

Theorem 2. Edge theorem (Bartlett, *et al*., 1988)
Let Q be a p-dimensional polytope that is its vertices and edges describe the convex hull (17). Then the boundary of $R(Q)$ is contained in the root space of the exposed edges of Q.

□

Due to Theorem 2 the characteristic polynomials (16) will be stable if and only if the following set of segments are stable

$$E(s)=\{\lambda A_{Vi}(s)+(1-\lambda)A_{Vj}(s)\},\ i,j=1,2\ldots,p2^{p-1} \tag{18}$$

Both i and j has to be taken as the vertices number of corresponding edges.
Substituting (16) to (18) after some manipulation one obtains the following extremal transfer function for affine system (14)

$$G_P(s)=\frac{\lambda P_{V1,i}+(1-\lambda)P_{V1,j}}{\lambda P_{V2,i}+(1-\lambda)P_{V2,j}},\ \lambda\in\langle 0,1\rangle \tag{19}$$

Lemma 1.
The controller (2) stabilizes the affine system (14) for all $q \in Q$ if and only if the controller stabilizes the extremal transfer function for polytopic system (19) for all $\lambda \in \langle 0, 1\rangle$.

□

The problem addressed in the Theorem 1 deals with a polytope and therefore using the Edge theorem can solve it. In general, the sets of extremal transfer functions (9) and (19) are quite different. While the number of $G_E(s)$ is equal to 32 ($G_K(s)$ - 16) the number of extremal transfer functions (19) depends exponentially on the number of uncertain parameters q_i, $i = 1, 2, ..., p$ and equal to $p2^{p-1}$.

3.3 Multilinear case

Let the uncertain plant transfer function (1) be

$$G(s)=\frac{P_{11}(s)P_{12}(s)\cdots P_{1n}(s)}{P_{21}(s)P_{22}(s)\cdots P_{2d}(s)} \tag{20}$$

where $P_{ij}(s)=p_0^{ij}+p_1^{ij}s+\cdots+p_{n_{ij}}^{ij}s^{n_{ij}}$.

Each $P_{ij}(s)$, $i = 1, 2, ..., n(d)$ belongs to a linear interval polynomial specified as

$$p_k^{ij}\in\langle \underline{p}_k^{ij},\ \overline{p}_k^{ij}\rangle,\ i=1,2,\ j=0,1,\ldots,n(d),\ k=0,1,\ldots,n_{ij}$$

with independently varying parameters. Let $K_{ij}(s)$ and $S_{ij}(s)$ denote the respective Kharitonov polynomials and Kharitonov segments of corresponding real interval polynomial $P_{ij}(s)$.
An extremal transfer function is given as follows (Chapellat, *et al*., 1994)

$$M_E(s)=\left\{\frac{S_{11}(s)S_{12}(s)\cdots S_{1n}(s)}{K_{21}(s)K_{22}(s)\cdots K_{2d}(s)}\cup \frac{K_{11}(s)K_{12}(s)\cdots K_{1n}(s)}{S_{21}(s)S_{22}(s)\cdots S_{2d}(s)}\right\} \tag{21}$$

Lemma 2.
The controller (2) stabilizes the multilinear system (20) for the whole uncertainty box if and only if the controller stabilizes the extremal transfer function (21).

□

Note that the number of extremal transfer function of (21) is $2.4^d.4^n$.

Consider the transfer function of multilinear interval system to be a ratio of multilinear polynomials with independent parameters. A proper stable system with transfer function of the form (1) will be considered where

$$P_1(s)=\sum_{i=1}^{m}C_i(s)\prod_{j=1}^{r_i}D_{ij}(s) \qquad P_2(s)=\sum_{i=1}^{m}H_i(s)\prod_{j=1}^{k_i}L_{ij}(s) \tag{22}$$

with $C_i(s)$ and $H_i(s)$ being fixed polynomials and the $D_{ij}(s)$ and $L_{ij}(s)$ being independent real linear interval polynomials.
Let d and l denote the sets of coefficients of the corresponding interval polynomials. d and l vary in a prescribed uncertainty box $d \in \Lambda$ and $l \in \Pi$.
$P_1(s)$ and $P_2(s)$ are coprime polynomials over the box $Q = \Pi \times \Lambda$ and it is assumed that $P_2(s) \neq 0$ for all $l \in \Pi$ and $s = j\omega$, $\omega > 0$. Introduce the Kharitonov

polynomials and segments associated with $L_{ij}(s)$ and $D_{ij}(s)$, respectively.
The extremal transfer function of (22) is given as follows (Bhattacharyya, *et al.*, 1995)

$$M_{SE}(s)=\left\{\frac{\Gamma_K(s)}{\Omega_E(s)}\cup\frac{\Gamma_E(s)}{\Omega_K(s)}\right\} \tag{23}$$

where

$$\Gamma_E^l(s)=\bigcup_{l=1}^{m}\Gamma_E^l(s)$$
$$\Gamma_E^l(s)=\sum_{i=1}^{m}C_i(s)\prod_{j=1}^{r_i}N_{Lij}(s)+C_l(s)\prod_{j=1}^{r_e}T_{Llj}(s) \tag{24}$$
$$\Gamma_K(s)=\sum_{i=1}^{m}C_i(s)\prod_{j=1}^{k_i}N_{Lij}(s)$$

The same equations hold for $\Omega_E(s)$ and $\Omega_K(s)$ with polynomial $P_2(s)$.

Lemma 3.
The controller (2) stabilizes the multilinear system (22) for all (d, l) if and only if the controller stabilizes the extremal transfer function (23).

□

Now, consider a multilinear polytopic system where the entries of the uncertainty vector $q^T=[q_1,\ldots,q_p]$ in the transfer function (1) are in multilinear form.
The closed-loop characteristic polynomial is given as follows (4)

$$A(s)=F_1(s)P_1(s,q)+F_2(s)P_2(s,q) \tag{25}$$

Let the uncertain parameters $q_i\in\langle\underline{q_i},\overline{q_i}\rangle$, $i = 1,2,\ldots,p$ belong to a p-dimensional uncertain parameter box Q with $N=2^p$ vertices and $p2^{p-1}$ edges.
Denote the characteristic polynomials in corresponding vertices of Q as follows

$$A_v(s,q)=\{A(s):\ q_i=\underline{q}_i\ \ or\ \ q_i=\overline{q}_i,\ i=1,2,\cdots,p\}=\{v_1(s),\ldots,v_N(s)\} \tag{26}$$

Let $\Delta(s)$ denote the convex hull of the vertex polynomials $A_v(s,q)$

$$\Delta(s)=\sum_{i=1}^{N}\lambda_i v_i(s),\ \sum_{i=1}^{N}\lambda_i=1 \tag{27}$$

where $\lambda_i\in\langle 0,1\rangle$.
Under the assumptions that
a) for any $q\in Q$, the polynomials (25) and (26) are of the same degree,
b) for any $s=j\omega$, $\omega>0$, $\Delta(s)\neq 0$ in (27),
c) there exists at least one $q^*\in Q$ such that (25) is stable,

the characteristic polynomial (25) is stable if the convex hull (27) and equivalently the sets of characteristic polynomial edges

$$E(s)=\{\lambda v_i(s)+(1-\lambda)v_j(s):\ v_i(s),v_j(s)\in A_v(s,q)\} \tag{28}$$

are stable where $\lambda\in\langle 0,1\rangle$.
With respect to (25), the vertex characteristic polynomials of (26) can be rewritten as follows

$$v_i=F_1(s)v_{P1i}+F_2(s)v_{P2i},\ i=1,2,\ldots,N \tag{29}$$

where $v_{P1i}(s)$, $v_{P2i}(s)$ are the vertex polynomials of $P_1(s,q)$, $P_2(s,q)$, respectively.
A simple manipulation of the entries of $E(s)$ yields

$$E_K(s)=F_1(s)[\lambda v_{P1i}+(1-\lambda)v_{P2j}]+F_2(s)[\lambda v_{P1i}+(1-\lambda)v_{P2j}] \tag{30}$$

With respect to (30), the extremal transfer function of the multilinear polytopic system is as follows

$$M_{SP}(s)=\frac{\lambda v_{P1i}(s)+(1-\lambda)v_{P1j}(s)}{\lambda v_{P2i}(s)+(1-\lambda)v_{P2j}(s)} \tag{31}$$

where $i\neq j\ \ i,j=1,2,\ldots,\frac{2^p!}{2(2^p-2)!}$.
Note that the Mapping Theorem (Hollot, Xu, 1989) shows that the image set of a multilinear interval polynomial (25) is contained in the convex hull of the vertices of Q (26). A sufficient condition for the entire image set to exclude zero (Zero Exclusion Principle) is that the convex hull excludes zero. This suggests that stability of the multilinear set (25) can be guaranteed by solving the stability of the convex hull of the vertex polynomials (27).

Lemma 4.
The controller (2) stabilizes the multilinear polytopic system with characteristic polynomial (25) for all $q\in Q$ if the controller (2) stabilizes the extremal transfer function (31).

□

Note that if Q is not an axis parallel box or the dependency on parameters in the characteristic polynomial (25) is not multilinear, the above lemma does not hold.

4. EXAMPLES

Example 1

As a real example we have considered the problem of robust controller design to control the speed of two serially connected small DC motors.
Stator voltage U_1 is the input signal of the first motor which speed ω_1 is the is converted to voltage U_2 which is input of the second DC motor which output speed ω_2 is to be controlled.
The controlled process has been identified in three working points using the ARMAX model. The interval transfer function of the process is of the form

$$G(s)=\frac{P_1(s)}{P_2(s)}=\frac{p_{12}s^2+p_{11}s+p_{10}}{p_{22}s^2+p_{21}s+p_{20}} \tag{32}$$

where $p_{22}=1$, $p_{21}\in[1.83\ 2.25]$, $p_{20}\in[0.64\ 0.67]$, $p_{12}\in[0.011\ 0.025]$, $p_{11}\in[-0.58\ -0.32]$, $p_{10}\in[1.54\ 2]$.

First design method

The robust controller design has been carried out using the Nejmark D-curve method (Nejmark, 1978). The method is based on splitting the space of parameters to regions with an equal number of unstable roots of the characteristic equation. In this method the curve for *P*, *I*, *D* gain have been obtained.
Fig. 1 depicts the D-curve for choosing the integration coefficient *I* of PID controller.

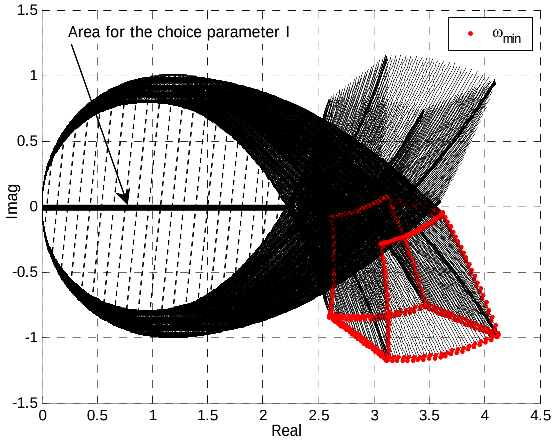

Fig. 1. D-curve

The robust PID controller designed using this method for 32 extremal transfer functions (9) is of the form

$$R(s) = P + \frac{I}{s} + Ds = 1.3 + \frac{0.6}{s} + 0.5s \qquad (33)$$

the degree of stability $\alpha = 0.633$ has been achieved.
Note: The degree of stability with a negative sign is equal to the maximum eigenvalue of the stable closed loop system defined by the extremal transfer function and a PID controller.

Second design method

The controller design is carried out based on the required settling time t_{reg}, phase margin $\Delta\varphi_0$ and maximum overshoot η_{max}. The method for determining η_{max} and t_{reg} from the open loop Bode diagrams was developed by Reinisch (Reinisch, 1974).
For all systems whose open loop transfer functions include integrator and are of the form

$$G_O(s) = \frac{K}{s}\frac{\prod_{j=1}^{m}(1+\tau_j s)}{\prod_{i=1}^{n-1}(1+\tau_i s)}, \quad m \le n \qquad (34)$$

Reinisch derived the following performance measures with a sufficient precision:
The relationship between the maximum overshoot η_{max} and $\varphi(\omega_0)$ is

$$\eta_{\max} = e^{-\frac{\pi}{\sqrt{-\frac{4\cos\varphi(\omega_0)}{\sin^2\varphi(\omega_0)}-1}}}.100, \quad \Delta\varphi_0 = 180 + \varphi(\omega_0) \qquad (35)$$

If we consider a dead zone δ = *5* [*%*] and the damping coefficient changing within the interval $b\in$ (0.25, 0.65) then

$$\frac{\pi}{\omega_0} < t_{reg} < 4\frac{\pi}{\omega_0} \qquad (36)$$

The inequality (36) specifies with a sufficient precision the settling time using the crossover frequency ω_0 at which the $G_0(j\omega)$ magnitude is equal 1.
Based on the required phase margin and using (35) and (36) we can design a robust controller from Bode diagrams.
The robust PID controller designed by this method for 32 extremal transfer functions (9) for settling time $t_{reg} = 5[s]$ and a phase margin $\Delta\varphi = 50[°]$ is of the form

$$R(s) = P + \frac{I}{s} + Ds = 2.07 + \frac{0.31}{s} + 0.73s \qquad (37)$$

We achieved the degree of stability $\alpha = 0.137$, settling time $t_{reg} = 4.7[s]$ and phase margin $\Delta\varphi = 44.1[°]$.

Third design method

The controller design is carried out by means of the criterion of the minimum of integral of the squared error (I_{SE}) and the criterion of minimum integral of squared error multiplied by time (I_{TSE}). The integral performance criterions provide information about the control process on the basis of integral error for all time values.
The algorithm for calculating I_{SE} designed by Nekolný (Nekolný, 1961) comes from the Parseval's integral in form

$$I_{SE} = \int_0^\infty [e(t)]^2 dt = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} E(s)E(-s)ds \qquad (38)$$

where *E(s)* is the Laplace transform of the tracking error

$$E(s) = \frac{D(s)}{A(s)} = \frac{d_{n-1}s^{n-1} + \cdots + d_1 s + d_0}{a_n s^n + \cdots + a_1 s + a_0} \qquad (39)$$

The integral of the square error multiplied by time can be written as

$$I_{TSE} = \int_0^\infty te^2(t)dt \qquad (40)$$

For the integral (40), formulas in the closed form have been derived.
Design of unknown controller parameters have been carried out using *minimax problem* formulation

$$\min_x \max_{\{F_i\}}\{F_i(x)\} \qquad (41)$$

which is realized by the *fminimax* function in the Matlab Optimization Toolbox. This function minimizes the worst-case value of a set of multivariable functions.
The robust PI controllers designed by the I_{SE} and I_{TSE} criterions, respectively, using extremal transfer functions $G_P(s)$ (19) are of the form

$$R(s) = P + \frac{I}{s} = 0.86 + \frac{0.24}{s} \quad (I_{SE}) \tag{42}$$

$$R(s) = P + \frac{I}{s} = 1.05 + \frac{0.23}{s} \quad (I_{TSE}) \tag{43}$$

We achieved the degree of stability $\alpha = 0.209$ for the I_{SE} criterion and $\alpha = 0.17$ for the I_{TSE} criterion.

Example 2

In this example the interval plant proposed by Hollot and Yang (1990) has been considered

$$G(s) = \frac{P_1(s)}{P_2(s)} = \frac{p_{10}}{(s+0.1)(s+0.2)(s+25)(s+75)} \tag{44}$$

where $p_{10} \in [1;\ 5000]$.
As it can be observed, this plant has only one uncertainty parameter.

Fourth design method

The robust controller design is carried out by Bode diagrams (Kuo, 1991). There are 404 extremal transfer functions derived for linear interval uncertainty. The Bode diagram for extremal transfer functions $G_P(s)$ (19) is shown in Fig. 2. In the frequency domain a robust PI controller for a required phase margin $\Delta\varphi = 50°$ has been designed in the form

$$R(s) = K\left(1 + \frac{1}{T_i s}\right) = \frac{0.0376s + 0.00104}{s} \tag{45}$$

where $K = 0.0376$ and $T_i = 36.2$ [s].

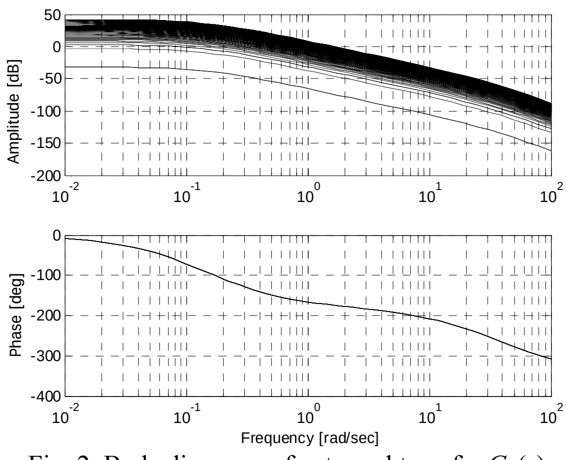

Fig. 2. Bode diagrams of extremal transfer $G_P(s)$

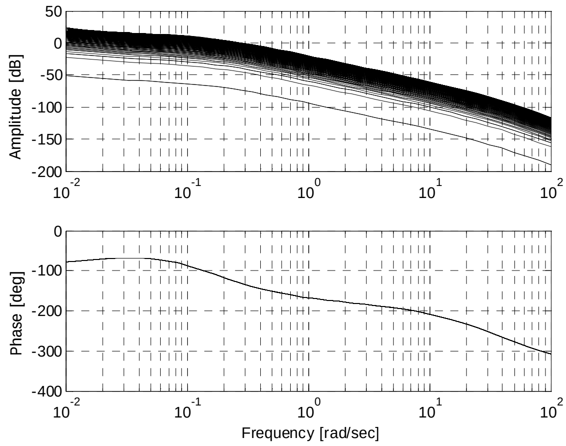

Fig. 3. Bode diagrams of the open loop system: $G_P(s).R(s)$

Fig. 3 depicts Bode diagrams of the open loop system. The achieved gain margin ΔK and phase margin $\Delta\varphi$ are:

$$\begin{aligned} \Delta K &= 34.2 \text{ dB} \\ \Delta\varphi &= 49.5° \end{aligned} \tag{46}$$

From the Bode diagrams it is possible to see that the designed robust controller guarantees stability and performance for linear interval transfer function with uncertainties.

Fifth design method

We will design a robust controller for a linear system (44) described by polynomial matrices (Henrion *et al.*, 2002). The design problem amounts to finding a dynamical output-feedback controller with a transfer function $F_2^{-1}(s)F_1(s)$ such that the closed-loop denominator matrix

$$A(s) = F_1(s)P_1(s) + F_2(s)P_2(s) \tag{47}$$

is robustly stable for all admissible uncertainties.
Let us assume that the static feedback matrix K satisfies structural LMI constraints and the controller polynomial matrices $F_1(s) = F_{10} + F_{11}s + \ldots$ and $F_2(s) = F_{20} + F_{21}s + \ldots$ entering linearly in polynomial matrix $A(s)$ have a prescribed structure, which we denote by the LMI

$$G(A) \geq 0. \tag{48}$$

For a PID controller the coefficients $F_1(s)$ and $F_2(s)$ will be

$$\frac{F_1(s)}{F_2(s)} = K_P + \frac{K_I}{s} + K_D s \tag{49}$$

where $F_{20} = 0$, $F_{21} = 1$, $F_{22} = 0$ and $F_{10} = K_I$, $F_{11} = K_P$, $F_{12} = K_D$.
Under these assumptions the following lemma (Henrion *et al.*, 2002) can be formulated

Lemma 5.
The transfer function in the affine form (14) with uncertainty q_i, $i = 1,\ldots, p$ is robustly stabilizable by a constrained output feedback controller $F_1(s)$, $F_2(s)$ if

for a stable nominal characteristic polynomial $D(s)$ of the same degree as polynomial matrices $A_i(s) = P_{1i}(s)F_1(s) + P_{2i}(s)F_2(s)$, there exist some matrices $P_i = P_i^T$ satisfying the LMI

$$D^T A_i + A_i^T D - H(P_i) > 0,\ i = 1,\dots,m \tag{50}$$

with the additional LMI constraint (48).

□

<u>Note:</u> $H(P) = \Pi^T (H \otimes P)\Pi$ where H is Hermitian matrix $H = \begin{bmatrix} 2\alpha & 1 \\ 1 & 0 \end{bmatrix}$, α is degree of stability and a projection matrix

$$\Pi = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}.$$

In our case a nominal controller is $R_n(s) = 0.05 + 0.002/s + 0.001s$ and with help of SeDuMi we have obtained the robustly stabilizing PID controller

$$R(s) = P + \frac{I}{s} + Ds = 0.71 + \frac{0.14}{s} + 2.54s \tag{51}$$

The achieved degree of stability is $\alpha = 0.003935$.

5. CONCLUSION

The main aim of this paper has been to present a survey of extremal transfer functions and robust controller design using classical control theory approach. The proposed robust controller design procedures with extremal transfer functions guarantee specificified performance and stability with necessary and sufficient or sufficient stability conditions.

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