ROBUST STABILITY ANALYSIS OF LINEAR SYSTEMS

Ľ. Grman, V. Veselý

Department of Automatic Control Systems,
Faculty of Electrical Engineering and Information Technology,
Slovak University of Technology
Ilkovičova 3, 812 19 Bratislava, Slovak Republic
fax : +421-2-65429734 and e-mail: grman@kasr.elf.stuba.sk

Abstract: In this paper, several numerical experiments are performed in order to compare five methods based on linear matrix inequalities conditions and parameter-dependent Lyapunov functions for continuous-time systems. Numerical examples thoroughly illustrate the power of the considered robust stability analysis methods and show which one provides the less conservative results.

Keywords: Robust stability, Lyapunov function, LMI condition.

1 INTRODUCTION

Robustness has been recognized as a key issue in the analysis and design of control systems during the last two decades. The field of robust control methods with small-gain-like robustness conditions started with the pioneering work of Zames (Zames 1981) where the consequence of the robust control paradigm yielded the definition of the control design problem as an optimization problem. Only at the end of 80's a practical solution to this problem was found in (Doyle 1989, Fans 1991).

This paper belong to the Lyapunov class of methods and LMI one (Boyd *et al*. 1994). The convex polytope-type uncertainty description of uncertain systems found a natural framework for it's accounting in the LMI's formalism. The basis for LMI stability analysis condition of such system is termed by quadratic stability. A weakness of quadratic stability is that it guards against arbitrary fast parameter variations and therefore is based on the use of a single Lyapunov function for testing stability over the whole uncertainty box. To reduce conservativism of quadratic stability for polytopic system the parameter dependent Lyapunov function has been introduced (Gahinet *et al.* 1996, Henrion *et al.* 2002, Takahashi *et al.* 2002) for the robust stability analysis of continuous systems.

This work deals with the comparison of numerical results obtained from the robust stability analysis with different parameter-dependent Lyapunov functions and continuous-time systems.

2 PROBLEM FORMULATION AND PRELIMINARIES

This paper is concerned with the class of uncertain linear system that can be described as

$$\delta(x) = \left(A_0 + A_1\Theta_1 + A_2\Theta_2 + \cdots + A_p\Theta_p\right)x = A(\Theta)x \quad (1)$$

where $x \in R^n$, $\Theta = \left[\Theta_1 \ldots \Theta_p\right] \in R^p$ is the state vector and vector of uncertain and possible time varying parameters, $\delta(.)$ denotes the derivative operator,

$$\Theta_j \in \left\langle \underline{\Theta}_j, \overline{\Theta}_j \right\rangle,\ \dot{\Theta}_j \in \left\langle \underline{r}_j, \overline{r}_j \right\rangle,\ j = 1, 2, \ldots, p \quad (2)$$

where $\underline{\Theta}_j, \overline{\Theta}_j, \underline{r}_j, \overline{r}_j$ are known lower and upper uncertainty bounds. Let Γ and Λ denote the sets of corners of the parameter box (2) and of the rate of variation box (2), respectively

$$\Gamma = \left\{\left(\gamma_1, \ldots, \gamma_p\right) : \gamma_j \in \left\langle \underline{\Theta}_j, \overline{\Theta}_j \right\rangle\right\}$$

$$\Lambda = \left\{\left(\lambda_1, \ldots, \lambda_p\right) : \lambda_j \in \left\langle \underline{r}_j, \overline{r}_j \right\rangle\right\} \quad (3)$$

and let

$$\Theta_m = \left[\frac{\underline{\Theta}_1 + \overline{\Theta}_1}{2}, \dots, \frac{\underline{\Theta}_p + \overline{\Theta}_p}{2}\right] \qquad (4)$$

denote the average value of the vector uncertain parameters.

The system represented by (1) is a polytope of linear affine systems, which can be described by a list of its vertices

$$\delta(x) = A_{vi}x, \qquad i = 1, 2, \ldots, N \qquad (5)$$

where $N = 2^p$.

The linear uncertain system (5) belongs to a convex polytopic set defined as

$$\delta(x) = A(\alpha)x \qquad (6)$$

where

$$S := \left\{A(\alpha) : A(\alpha) = \sum_{i=1}^{N} \alpha_i A_{vi},\ \sum_{i=1}^{N} \alpha_i = 1,\ \alpha_i \geq 0\right\} \qquad (7)$$

Using Lyapunov stability it is possible to give the following definition.

Definition 1:

System (6) is robustly stable in the uncertainty domain (7) if and only if there exists a matrix $P(\alpha) = P(\alpha)^T > 0$ such that

$$A(\alpha)^T P(\alpha) + P(\alpha)A(\alpha) < 0, \qquad (8)$$

for all α such that $A(\alpha) \in S$.

□

There is no general and systematic way to formally determine $P(\alpha)$ as a function of $A(\alpha)$ and uncertain parameter α. Such a matrix $P(\alpha)$ is called a parameter-dependent Lyapunov matrix and for concrete structure of $P(\alpha)$ the inequality (8) defines the parameter dependent quadratic stability (PDQS). Effective way to choose $P(\alpha)$ is to look for a single Lyapunov matrix $P(\alpha) = P$. The latter case is analyzed in the following lemma.

Lemma 1:

Uncertain system (6) is robustly quadratically stable in the uncertain domain (7) if and only if there exists a matrix $P = P^T > 0$ such that

$$A_{vi}^T P + PA_{vi} < 0, \qquad (9)$$

for all $i = 1, 2, \ldots, N$.

□

Unfortunately, this approach generally provides quite conservative results. To reduce the conservativism when (1) is affine in Θ and the system parameters are time invariant, in (Gahinet *et al.* 1996) the parameter-dependent Lyapunov function $P(\Theta)$ has been used in the form

$$P(\Theta) = P_0 + P_1\Theta_1 + P_2\Theta_2 + \cdots + P_p\Theta_p \qquad (10)$$

Due to (Gahinet *et al.* 1996) the sufficient affine quadratic stability conditions are given by the next lemma.

Lemma 2:

Consider the linear systems governed by (1) and parameter-dependent Lyapunov function (10). The continuous-time system (1) is affine quadratically stable if $A(\Theta_m)$ is stable and there exist p+1 symmetric matrices $P_0, P_1, \ldots, P_p$ such that $P(\Theta) > 0$ satisfies

$$\begin{aligned} L(\gamma, \lambda) = {} & A(\Theta)^T P(\Theta) + P(\Theta)A(\Theta) + \\ & + P(\lambda) - P_0 + \sum_{j=1}^{p} \Theta_j^2 M_j < 0 \end{aligned} \qquad (11)$$

for all $(\gamma \times \lambda) \in \Gamma \times \Lambda$ and

$$A_j^T P_j + P_j A_j + M_j \geq 0 \qquad (12)$$

for all $j = 1, 2, \ldots, p$, where $M_j = M_j^T \geq 0$ are some nonnegative definite matrices.

□

Affine quadratic stability encompasses quadratic stability but it can be more conservative than other parameter-dependent quadratic stability. In (Henrion *et al.* 2002, Takahashi *et al.* 2002) and in this paper the following parameter-dependent Lyapunov matrix has been used

$$P(\alpha) = \sum_{i=1}^{N} P_i \alpha_i \qquad (13)$$

which has to be positive definite for all values of α such that $A(\alpha) \in S$.

Lemma 3: (Takahashi *et al.* 2002)

System (6) with parameter-dependent Lyapunov matrix (13) is PDQS if

$$A_{vi}^T P_i + P_i A_{vi} < -I, \quad P_i > 0, \quad i = 1, 2, \ldots, N \qquad (14)$$

$$A_{vk}^T P_j + P_j A_{vk} + A_{vj}^T P_k + P_k A_{vj} < \frac{2}{N-1} I, \qquad (15)$$

$$k = 1, 2, \ldots, N-1, \quad j = k+1, \ldots, N$$

where I is identity matrix.

□

The following robust stability analysis can be found in (Henrion *et al.* 2002).

Lemma 4:

System (6) with parameter-dependent Lyapunov matrix (13) is PDQS if there exists a matrix F and matrices $P_i = P_i^T > 0$ satisfying the LMI

$$\begin{bmatrix} F^T A_{vi} + A_{vi}^T F - aP_i & \left(A_{vi} + F + b^* P_i\right)^T \\ A_{vi} + F + b^* P_i & 2I - cP_i \end{bmatrix} > 0 \qquad (16)$$

i = 1, 2,..., N

where

$$D = \left\{ s \in C : \begin{bmatrix} 1 \\ s \end{bmatrix}^* \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} < 0 \right\} \qquad (17)$$

is a stability region in the complex plane and the star denotes transpose conjugate.

□

Standard choice for D is the left half-plane (a = 0, b = 1, c = 0). Obviously matrix F in (16) is stable.

In the paper (Yoshia Ebihara and Tomomichi Hagiwara 2002) the authors have proposed a general approach to the dilated characterizations in the continuous-time setting. The proposed method pursues the idea of (Henrion *et al.* 2002) to introduce a new auxiliary variable to achieve decoupling between the Lyapunov variables and the controller variables. These nice and interesting features enable us to deal with multiobjective and robust control for real polytopic uncertainty with the use of non-common and parameter-dependent Lyapunov function (13). The results of (Yoshia Ebihara and Tomomichi Hagiwara 2002) for robust stability analysis are summarized as follows.

Lemma 5

System (6) with parameter-dependent Lyapunov function (13) is PDQS if there exist $P_i = P_i^T > 0$ and a matrix G such that

$$\begin{bmatrix} P_i + \left(A_{vi} - 0.5I\right)^T G & -P_i - \left(A_{vi} - 0.5I\right)^T G + G^T \\ -P_i + G - G^T\left(A_{vi} - 0.5I\right) & -G - G^T \end{bmatrix} < 0 \qquad (18)$$

□

3 EXAMPLES

3.1 Method evaluation concept

In this section the qualities and power of particular methods provided in Sections 2 is tested on 500 random generated affine stable closed loop systems. To evaluate the conservativeness of the methods, we adopt "the size of stability region" for each tested example measured by the parameter q respective to maximal polytope of uncertainties for which the uncertain system remains stable. The motivation to the adopted approach can be given considering the robust control design task in the following interpretation.

Consider an uncertain affine linear system

$$\delta(x) = A(\Theta)x + B(\Theta)u \qquad (19)$$

where $u \in R^m$ is input vector

$$B(\Theta) = B_0 + B_1\Theta_1 + \cdots + B_p\Theta_p \in R^{n\times m}.$$

For the static output feedback

$$u = Ky = KCx \qquad (20)$$

where $y \in R^l$, $C \in R^{l\times n}$ is output variable and output matrix of linear system (19) respectively, the closed loop system of (19) is

$$\delta(x) = \left(A(\Theta) + B(\Theta)KC\right)x = A_c(\Theta x). \qquad (21)$$

In this case the close loop robust stability analysis problem can be extended and the question arises "how robust" the considered controller is:

What is the maximal range of uncertainty parameters so that the close loop affine uncertain system (21) remains stable?

$$q = \max|\Theta_1| = \max|\Theta_2| = \cdots = \max|\Theta_p|. \qquad (22)$$

In this section exhaustive numerical examples are given for evaluation and comparison of the above described five robust stability conditions. The 500 affine stable closed loop systems (21) were generated for $\Theta_j \in \langle -1; 1\rangle$, $j = 1, 2, ..., p$ and each pair (n = 3, p = 2), (n = 5, p = 2) and (n = 5, p = 3). Maximal value of uncertain parameter q respective to each of the considered robust stability conditions was computed for each example.

The summary comparison of the considered methods is provided in accordance with the following criteria: The mean value of uncertainty parameter q_m reached in considered 500 examples was computed for each method and the methods were ordered with the decreasing value of q_m.

3.2 Generated examples

For a case 500 affine stable closed loop system were generated for the following pairs (n = 3, p = 2), (n = 5, p = 2) and (n = 5, p = 3). The results are evaluated according to the concept explained in Section 3.1.

For the defined pairs the obtained results are summarized in the Table 1 and the corresponding Fig. 1.

Table 1

p	n		AQ	HEN	EBI	TAKA	Q
2	3	Place index	1.02	1.64	2.41	3.67	3.84
		q_m	3.02	2.91	2.67	2.95	2.32
		σ_S	1.66	1.56	1.25	1.62	1.36
	5	Place index	1.02	1.70	2.62	3.41	4.44
		q_m	1.96	1.86	1.73	1.89	1.35
		σ_S	0.91	0.85	0.67	0.91	0.70
3	5	Place index	1.00	1.88	2.96	3.08	4.61
		q_m	1.61	1.55	1.43	1.57	1.09
		σ_S	0.57	0.52	0.38	0.56	0.46

where the above acronyms read as follows

- AQ – affine quadratic stability (Gahinet *et al.* 1996), *Lemma 2*
- EBI – Parameter dependent quadratic stability proposed in (Yoshia Ebihara and Tomomichi Hagiwara 2002), *Lemma 5*
- HEN – Parameter dependent quadratic stability proposed in (Henrion *et al.* 2002), *Lemma 4*
- Q – quadratic stability, see (Boyd *et al.* 1994), *Lemma 1*
- TAKA – Parameter dependent quadratic stability proposed in (Takahashi *et al.* 2002), *Lemma 3*

The place index, mean value q_m of the uncertainty q and standard deviation σ_S correspond to the mean values obtained from 500 robust stability analysis calculations. Note that for particular affine system if the concrete method took the first place it has got one point, if two methods took first place both have got one point and then the next method has got 3 points, etc.

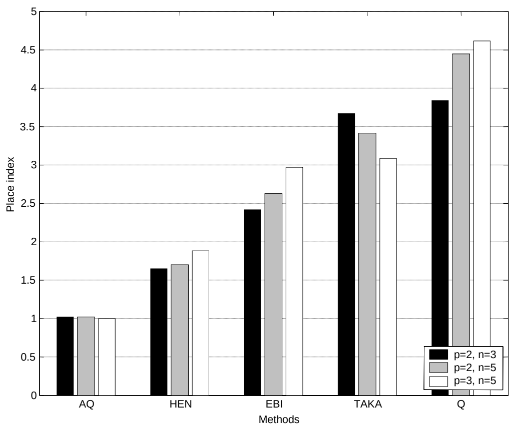

Fig. 1 – The result of robust stability calculation for value of place index

4 CONCLUSION

This paper has presented a numerical comparison among five methods based on LMI conditions and parameter-dependent Lyapunov functions. The results show that the affine quadratic criterion is undoubtedly the less conservative for all three cases. When the affine system becomes more complex (n and p increases) quadratic stability criterion moves to lower place, the place index changes from 3.84 to 4.61. Criterion TAKA moves in the opposite direction, the place index changes from 3.67 to 3.08. Criterions Q, HEN, EBI become more conservative when the affine system becomes more complex.

ACKNOWLEDGMENTS

The work has been supported by the Slovak Scientific Grant Agency, Grant No. 1/0158/03.

5 REFERENCES

Boyd, S., El Ghaoni, L., Feron, E., Balakrishnon V. (1994). Linear Matrix Inequalities in System and Control Theory. *SIAM Studies in Applied Mathematics.* Philadelphia.

Doyle, J.C., Glover, K., Khargoneker, P. P., Francis, B. (1989). State space solutions to the standard H_2 and H_∞ control problems. *IEEE Transactions on Circuits and Systems*, 34, (8), 831-847.

Fans, M. K. H., Tits, A. L., Doyle, J. C. (1991). Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics. *IEEE Transactions on Circuits and Systems*, 36, (1), 25-38.

Gahinet, P., Aspankarin, P., Chilali M. (1996). Affine Parameter-Dependent Lyapunov Functions and Real Parametric Uncertainty, *IEEE Transactions on Circuits and Systems*, 41, (3), 436-442.

Henrion, D., Arzelier D., Peaucelle D. (2002). Positive Polynomial Matrices and Improved LMI Robustness Conditions, In: *15th Triennial World Congress of the International Federation of Automatic Control*, Barcelona, CD.

Takahashi, R. H. C., Ramos, D. C. W., Peres, P. L. D. (2002). Robust Control Synthesis via a Genetic Algorithm and LMI's, In: *15th Triennial World Congress of the International Federation of Automatic Control*, Barcelona, CD.

Yoshio Ebihara, Tomomichi Hagiwara (2002). New Dilated LMI Characterizations for Continuous-Time Control Design and Robust Multiobjective Control. In: *Proceedings of the American Control Conference*, Anchorage, AK, 47-52.

Zames, G. (1981). Feedback and optimal sensitivity: model reference transformations, multiplicative semi norms and approximate inverses. *IEEE Transactions on Circuits and Systems*, 26, (2), 301-320.